Acta Mech
https://doi.org/10.1007/s00707-020-02702-x

ORIGINAL PAPER

Sandipan Paul · Alan D. Freed

A simple and practical representation of compatibility condition derived using a QR decomposition of the deformation gradient

Received: 11 November 2019 / Revised: 30 March 2020

Abstract This paper examines a condition for the existence and uniqueness of a finite deformation field whenever a Gram–Schmidt (**QR**) factorization of the deformation gradient $\mathbf{F}$ is used. First, a compatibility condition is derived, provided that a right Cauchy–Green tensor $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ is prescribed. It is well-known that under this condition a vanishing of the Riemann curvature tensor $\mathbb{R}$ ensures compatibility of a finite deformation field. We derive a restriction imposed on Laplace stretch $\boldsymbol{\mathcal{U}}$, arising from a **QR** decomposition of the deformation gradient, through this compatibility condition. The derived condition on Laplace stretch is unambiguous, because a Cholesky factorization of the right Cauchy–Green tensor ensures the existence of a unique Laplace stretch. Although a vanishing of the Riemann curvature tensor provides a necessary and sufficient compatibility condition from a purely geometric point of view, this condition lacks a direct physical interpretation in a sense that one cannot identify the restrictions imposed by this condition on a quantity that can be readily measured from experiments. On the other hand, our compatibility condition restricts dependence of components of a Laplace stretch on certain spatial variables in a reference configuration. Unlike the symmetric right Cauchy–Green stretch tensor $\mathbf{U}$ obtained from a traditional polar decomposition of $\mathbf{F}$, the components of Laplace stretch can be measured from experiments. Thus, this newly derived compatibility condition provides a physical meaning to the somewhat abstract idea of the traditionally used compatibility condition, viz., a vanishing of the Riemann curvature tensor. Couplings between certain components of the Laplace stretch representing shear and elongation play a crucial role in deriving this condition. Finally, implications of this compatibility condition are discussed.

1 Introduction

The objective of this paper is to determine a compatibility condition in terms of the Laplace stretch [8], viz., $\boldsymbol{\mathcal{U}}$, for a preassigned right Cauchy–Green tensor, $\mathbf{C} = \mathbf{F}^T\mathbf{F}$. Laplace stretch is an upper-triangular matrix arising from a Gram–Schmidt factorization of the deformation gradient, i.e., $\mathbf{F} = \boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{U}}$ where $\boldsymbol{\mathcal{R}}$ is orthogonal. The conditions for integrability derived herein restrict a dependence of components for $\boldsymbol{\mathcal{U}}$ on the spatial variables. Moreover, these conditions are physically significant due to the physical meaning of the components of Laplace stretch that can be readily measured from experiments.

The problem of existence and uniqueness of a finite deformation generating a prescribed right Cauchy–Green tensor has been addressed many times in the past century. The issue is mathematically equivalent to determining a condition such that a prescribed, symmetric, second-order tensor acts as the metric of an

S. Paul · A. D. Freed (✉)
Department of Mechanical Engineering, Texas A&M University, College Station, TX 77843, USA
E-mail: afreed@tamu.edu

A. D. Freed
Impact Physics Branch, U.S. Army Research Laboratory, Aberdeen Proving Ground, Aberdeen, MD 21005, USA

Published online: 09 June 2020

Euclidean space. This condition is provided by a popular theorem, first asserted by Riemann (1854) without proof. Later, in his Paris prize essay (1861), he proved necessity and stated that sufficiency is not hard to prove. (For a modern proof of this theorem, cf. Veblen [25], Sokolnikoff [20], Spivak [21] and Clayton [6]). According to this theorem, vanishing of the Riemann curvature tensor ensures that the right Cauchy–Green tensor **C** is the metric tensor for a Lagrangian frame of reference and, hence, it is possible to obtain a deformation map (or displacement vector) by integrating a system of partial differential equations involving **C** and the deformation map (or displacement vector).

Many forms of the Riemann curvature tensor have been derived. A list of other works on this topic is given in Truesdell and Toupin [24]. However, this compatibility condition does not involve any decomposition of the deformation gradient. Shield [19] employed a polar decomposition of the deformation gradient in order to obtain an integrability condition for the rotation tensor. He showed that the fourth-order tensor corresponding to his integrability condition and Riemann's curvature tensor are related through the inverse of a symmetric, stretch tensor arising from a polar decomposition of **F**. Positive-definiteness of the stretch tensor (hence, it is always invertible) ensures uniqueness in his relation. Thus, the integrability condition for a rotation tensor is equivalent to a vanishing of the Riemann curvature tensor. Blume [3] and Acharya [1] determined compatibility conditions in terms of the left Cauchy–Green tensor, i.e., $\mathbf{B} = \mathbf{F}\mathbf{F}^T$, for plane and three-dimensional deformations, respectively. In all these works, the body undergoing deformation is considered to be simply-connected. Yavari [26] studied the compatibility condition for a non-simply connected body from a geometric point of view.

A well adopted technique in the mathematics literature is the decomposition of a matrix with positive determinant into a product between a proper orthogonal matrix and an upper-triangular matrix that aligns with a specific set of orthogonal base vectors. This technique is commonly known as the Gram–Schmidt factorization of matrices. The idea of this procedure was first introduced by Laplace (1820) where he employed successive orthogonal projections to solve a least squares problem to estimate the masses of Jupiter and Saturn. Gram (1879) used this technique in his work on series expansions of real functions. This algorithm became popular when Schmidt (1907) used this technique to solve integral equations. Although the techniques in these pioneering works are essentially the same, their algorithms are different. A review of Gram–Schmidt factorization can be found in Leon et al. [15].

McLellan [16] introduced this technique into the physics literature. He was the first to apply a Gram–Schmidt factorization to the matrix of the deformation gradient in a chosen basis. This decomposition requires specification of a coordinate system. An important feature of this coordinate system is that the 1 coordinate direction and the normal to 12 coordinate plane remain invariant under a transformation of the upper-triangular matrix, $\boldsymbol{\mathcal{U}}$.[1] This is due to the fact that the Gram–Schmidt process requires a specification of these 1 coordinate direction and 12 coordinate plane prior to finding other coordinate directions through Laplace's successive orthogonal projection. A strategy to choose a unique coordinate system based on the deformation of the body has been proposed by Paul et al. [18]. McLellan [17] also showed that this decomposition has an added advantage over the classical polar decomposition, as upper-triangular matrices with positive determinant form a group under multiplication. This feature allowed him to further decompose the upper-triangular matrix $\boldsymbol{\mathcal{U}}$ into a diagonal matrix whose diagonal elements represent elongations along the coordinate axes, and an unit upper-triangular matrix whose off-diagonal elements represent three simple shears acting perpendicular to one another, thereby resulting in an Iwasawa [11] matrix decomposition of the deformation gradient. McLellan applied this decomposition in his work on the thermodynamic stability of crystalline phases. Srinivasa [22] showed that this decomposition has some more advantages over the classical polar decomposition of **F**. Of which, the most important is the direct physical meaning of the components of $\boldsymbol{\mathcal{U}}$,[2] and hence its utility regarding experiments. The coordinate frame in which a **QR** decomposition of the deformation gradient is performed, when aligned with a laboratory apparatus, enables one to find components $\mathcal{U}_{ij}$ unambiguously from experiments. Therefore, this coordinate frame has been termed an experimenter's frame of reference [7]. **QR** kinematics have been further explored by Freed and Srinivasa [9].

Lembo [14] investigated the compatibility condition for $\boldsymbol{\mathcal{U}}$. In his paper, he adopted a procedure similar to that of Shield's [19]; Lembo employed the technique of Shield to a **QR** decomposition of **F**, viz., $\mathbf{F} = \boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{U}}$ in our notation. Lembo considered the rotation tensor to be the primary variable and then determined a partial differential equation that solves for the rotation tensor $\boldsymbol{\mathcal{R}}$. He showed that the integrability condition for this partial differential equation is equivalent to a vanishing of the Riemann curvature tensor. Finally, a compatibility condition for Laplace stretch was obtained from the integrability of its deformation gradient. However, for

[1] McLellan denoted this upper-triangular matrix as **H**.

[2] Srinavasa denoted this upper-triangular matrix as $\tilde{\mathbf{F}}$.

a prescribed right Cauchy–Green tensor, a deformation gradient can only be obtained within a rigid body rotation. This indeterminacy is resolved by imposing a boundary condition on the system of PDEs. Here, we choose the Cauchy–Green tensor **C** to be the fundamental variable (cf. [2,5]) to derive a compatibility condition without any possible indeterminacy on **F**. Also, we show that Lembo's compatibility condition is an alternative statement for symmetry of the Christoffel symbol, which is obvious in view of the symmetry of the right Cauchy–Green tensor.

In this paper, we examine the condition imposed on $\boldsymbol{\mathcal{U}}$ by addressing the question of existence and uniqueness of a finite deformation generating a prescribed right Cauchy–Green tensor (or Laplace stretch). Considering **C** to be the primary variable, we seek a compatibility condition without involving the deformation gradient in the process. The final outcome of this work is significant, and vastly different from Lembo's compatibility condition. Moreover, the newly derived compatibility condition provides a physical meaning to the somewhat abstract and purely geometric condition of a vanishing of the Riemann curvature tensor, hitherto used to ensure compatibility of a finite deformation field for a prescribed right Cauchy–Green tensor.

The paper is organized as follows. First, we review the **QR** decomposition of a deformation gradient, and the kinematics pertinent to such a decomposition. This is followed by a brief review of Lembo's work on compatibility. Next, a detailed derivation of the compatibility condition for $\boldsymbol{\mathcal{U}}$ is provided, given a prescribed right Cauchy–Green tensor. This condition is comprised of five equations that need to be satisfied, which arise because of specified couplings between three orthogonal shears with two orthogonal elongations. These couplings are not arbitrary; they are very specific; they are a consequence of Gram–Schmidt factorization. Finally, we discuss implications of this compatibility condition. In this paper, we represent scalars with lowercase Roman/Greek letters, vectors with uppercase Roman/Greek letters in boldface and italics,[3] while second-,[4] third- and fourth-order tensors are represented with Roman letters in boldface, typewriter font and blackboard font, respectively.

2 Gram–Schmidt factorization of deformation gradient

Consider a simply connected body $\mathcal{B}$ embedded in a three-dimensional Euclidean point space. The undeformed configuration of the body is denoted by $\kappa_r(\mathcal{B})$. The motion $\mathcal{X}(\mathbf{X}, t)$ maps points in an undeformed configuration $\kappa_r(\mathcal{B})$ into points in a current configuration $\kappa_t(\mathcal{B})$. Position vectors of a material point in the undeformed and current configurations are denoted by $\boldsymbol{X}$ and $\boldsymbol{x}$, respectively. An assumption of simple-connectedness of the body ensures the applicability of Stokes' theorem. The deformation gradient **F** is a linear transformation that maps tangent vectors at a point in $\kappa_r(\mathcal{B})$ into tangent vectors at its corresponding point in $\kappa_t(\mathcal{B})$. We choose a Cartesian coordinate system $\{\tilde{\boldsymbol{e}}_I\}$ that aligns with our laboratory apparatus. In this coordinate system, one can decompose the matrix of **F** into an orthogonal matrix $\boldsymbol{\mathcal{R}}$ and an upper-triangular matrix $\boldsymbol{\mathcal{U}}$. $\boldsymbol{\mathcal{U}}$ is called Laplace stretch [8] and, in matrix form, can be written as:

$$[\mathcal{U}_{ij}] = \begin{bmatrix} a & \gamma & \beta \\ 0 & b & \alpha \\ 0 & 0 & c \end{bmatrix}, \tag{1}$$

where a, b, c are three, independent extensions along the coordinate axes, and $\gamma/a, \beta/a, \alpha/b$ represent three, independent shears acting perpendicular to each other. Note that a, b, c are positive, whereas α, β, γ can be positive, zero or negative.

This Laplace stretch can be decomposed further into a diagonal matrix and a unit upper-triangular matrix [9]. This is a direct consequence of the Iwasawa matrix decomposition of a deformation gradient [10,11]:

$$[\mathcal{U}_{ij}] = \begin{bmatrix} a & \gamma & \beta \\ 0 & b & \alpha \\ 0 & 0 & c \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} 1 & \gamma/a & \beta/a \\ 0 & 1 & \alpha/b \\ 0 & 0 & 1 \end{bmatrix}. \tag{2}$$

[3] Except for the current position vector $\boldsymbol{x}$, which retains its classical representation.

[4] Except for the Laplace stretch $\boldsymbol{\mathcal{U}}$ that is written in a bold calligraphic font to distinguish it from the classic, symmetric stretch **U** that arises from a polar decomposition of the deformation gradient, viz., $\mathbf{F} = \mathbf{RU}$ where **R** is orthogonal, $\mathbf{R} \neq \boldsymbol{\mathcal{R}}$.

The inverse of Laplace stretch is readily available and is written as:

$$[\mathcal{U}_{ij}^{-1}] = \begin{bmatrix} \dfrac{1}{a} & -\dfrac{\gamma}{ab} & -\dfrac{\alpha\gamma - b\beta}{abc} \\ 0 & \dfrac{1}{b} & -\dfrac{\alpha}{bc} \\ 0 & 0 & \dfrac{1}{c} \end{bmatrix}. \tag{3}$$

The right Cauchy–Green tensor $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ is related to Laplace stretch through

$$\mathbf{C} = \boldsymbol{\mathcal{U}}^T \boldsymbol{\mathcal{U}}. \tag{4}$$

Srinivasa [22] showed that it is possible to determine a unique $\boldsymbol{\mathcal{U}}$ generated from a given Cauchy–Green tensor $\mathbf{C}$ through its Cholesky factorization. The Laplace stretch, written in terms of the given components $\mathbf{C}_{ij}$ for the right Cauchy–Green tensor, becomes

$$[\mathcal{U}_{ij}] = \begin{bmatrix} \sqrt{C_{11}} & \dfrac{C_{12}}{\mathcal{U}_{11}} & \dfrac{C_{13}}{\mathcal{U}_{11}} \\ 0 & \sqrt{C_{22} - \mathcal{U}_{12}^2} & \dfrac{C_{23} - \mathcal{U}_{12}\mathcal{U}_{13}}{\mathcal{U}_{22}} \\ 0 & 0 & \sqrt{C_{33} - \mathcal{U}_{13}^2 - \mathcal{U}_{23}^2} \end{bmatrix}. \tag{5}$$

This Cholesky factorization proves that a Laplace stretch tensor can be *uniquely* determined from a right Cauchy–Green tensor.

3 Compatibility condition for Laplace stretch

Before going to the detailed derivation of our compatibility condition, we would like to briefly revisit Lembo's work on a similar topic, and to touch upon the issues that motivated us to delve deeper into it. In this work, a compatibility condition is obtained for a prescribed Cauchy–Green tensor. Because the base vectors $\{\tilde{\boldsymbol{e}}_I\}$ of the coordinate system form a Lagrangian triad in which the Gram–Schmidt factorization of $\mathbf{F}$ is to be performed, $\mathbf{C}$ is the metric of this coordinate system. For a deformation of the body to be compatible, its current configuration $\kappa_t(\mathcal{B})$ must be a Euclidean space whenever the undeformed configuration $\kappa_r(\mathcal{B})$ is Euclidean. A material manifold is considered to be Euclidean or flat when it is equipped with a metric-compatible, torsionless connection and the associated Riemann curvature tensor vanishes.

Since the torsion of the space $\kappa_t(\mathcal{B})$ vanishes, the connection coefficient, commonly known as the Christoffel symbol, ought to be symmetric. Therefore, the Christoffel symbol pertaining to this coordinate system takes the form

$$\mathsf{G}_{ijk} = \frac{1}{2}\left(C_{jk,i} + C_{ik,j} - C_{ij,k}\right), \tag{6}$$

where $C_{ij,k} = \partial C_{ij}/\partial X_k$, etc. The Riemann curvature tensor for this coordinate system is the same as the one wherein a polar decomposition is performed, because its definition does not involve any decomposition of $\mathbf{F}$. Hence, the Riemann curvature tensor is defined as

$$\mathbb{R}_{ijkl} = \mathsf{G}_{jli,k} - \mathsf{G}_{jki,l} + C_{pq}^{-1}\left(\mathsf{G}_{jkp}\mathsf{G}_{ilq} - \mathsf{G}_{jlp}\mathsf{G}_{ikq}\right), \tag{7}$$

where $i, j, k, l = 1, 2, 3$ and where repeated subscripts are summed according to Einstein's summation convention.

3.1 Review of previous work on a compatibility condition for $\boldsymbol{\mathcal{U}}$

Lembo [13] uses Burgatti [4]'s findings, where a rotation tensor is obtained from the symmetric stretch tensor $\mathbf{U}$ arising from polar decomposition of $\mathbf{F}$ [13]. For a prescribed right Cauchy–Green tensor, the stretch tensor $\mathbf{U}$ is easily obtained by using the relation $\mathbf{C} = \mathbf{U}^2$ and the symmetries of $\mathbf{C}$ and $\mathbf{U}$. Therefore, Burgatti's work allows one to find a deformation gradient whenever the right Cauchy–Green tensor is prescribed. However, this deformation gradient can only be determined up to a rigid body rotation.

With an assumption ensuring the existence of a deformation gradient, it immediately follows that $\mathrm{curl}(\mathbf{F}) = \mathbf{0}$, cf. e.g., (Clayton [6, §3.1.6]), because $\mathbf{F}$ is the gradient of a deformation map $\mathcal{X}$. Substitution of this condition into the definition for Christoffel symbol $\mathtt{G}$ yields

$$\mathtt{G}_{ikr} = F_{mi,k} F_{mr}. \tag{8}$$

Adopting a $\mathbf{QR}$ decomposition of $\mathbf{F}$, viz., $\mathbf{F} = \boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{U}}$, leads to a partial differential equation for the rotation $\boldsymbol{\mathcal{R}}$:

$$\frac{\partial \boldsymbol{\mathcal{R}}}{\partial \boldsymbol{X}} = \boldsymbol{\mathcal{R}}\mathtt{Z}, \tag{9}$$

where

$$\mathtt{Z}_{pnk} = \left[\mathtt{G}_{ikr}\mathcal{U}_{rp}^{-1} - \mathcal{U}_{pi,k}\right]\mathcal{U}_{in}^{-1}. \tag{10}$$

The integrability condition for this equation is [23]

$$\mathbb{Z}_{pnkl} = \mathtt{Z}_{pnk,l} - \mathtt{Z}_{pnl,k} + \mathtt{Z}_{snk}\mathtt{Z}_{psl} - \mathtt{Z}_{snl}\mathtt{Z}_{psk} = 0. \tag{11}$$

The fourth-order tensor $\mathbb{Z}$ is related to the Riemann curvature tensor through the inverse of Laplace stretch. Therefore, the integrability condition is equivalent to a vanishing of the Riemann curvature tensor. This relationship is unique because $\boldsymbol{\mathcal{U}}$ is always invertible. Because the rotation tensor belongs to the special orthogonal group $SO(3)$, i.e., the group of all orthogonal matrices with determinant $+1$, the partial differential equation for $\boldsymbol{\mathcal{R}}$ is solved by transforming it to its Lie algebra and then transferring it back to its Lie group. Finally, the author concluded that the compatibility condition for $\boldsymbol{\mathcal{U}}$ is given as:

$$\mathrm{curl}(\mathcal{U}_{ij}\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j) = \mathcal{U}_{mi}\,\mathtt{Z}^k_{mj}\,\boldsymbol{e}_k \times \boldsymbol{e}_i \otimes \boldsymbol{e}_j. \tag{12}$$

Clearly, Eq. (12) is derived from $\mathrm{curl}(\mathbf{F}) = \mathbf{0}$, which is the necessary and sufficient condition for existence and uniqueness of a finite deformation whenever a deformation gradient is prescribed. In fact, this condition is implicitly used in the derivation of Eq. (9), as shown before.

However, for a prescribed Cauchy–Green tensor, it is possible to find a deformation gradient only up to a rigid body rotation from the relation $\mathbf{C} = \mathbf{F}^T\mathbf{F}$. Blume [3] has shown that if two different deformation maps $\mathcal{X}$ and $\mathcal{X}'$ taken from the same undeformed configuration $\mathcal{B}$ generate the same right Cauchy–Green tensor, i.e.,

$$\left(\frac{\partial \boldsymbol{\mathcal{X}}}{\partial \boldsymbol{X}}\right)^T\left(\frac{\partial \boldsymbol{\mathcal{X}}}{\partial \boldsymbol{X}}\right) = \left(\frac{\partial \boldsymbol{\mathcal{X}}'}{\partial \boldsymbol{X}}\right)^T\left(\frac{\partial \boldsymbol{\mathcal{X}}'}{\partial \boldsymbol{X}}\right) = \mathbf{C}, \tag{13}$$

then $\mathcal{X}$ and $\mathcal{X}'$ must be related by

$$\boldsymbol{\mathcal{X}}'(\boldsymbol{X}, t) = \mathbf{Q}(t)\boldsymbol{\mathcal{X}}(\boldsymbol{X}, t) + \boldsymbol{d}(t), \tag{14}$$

where $\mathbf{Q}(t)$ is any arbitrary proper orthogonal tensor and $\boldsymbol{d}(t)$ is a vector. Therefore, the deformation gradients $\mathbf{F}$ and $\mathbf{F}'$ associated with these deformation maps $\boldsymbol{\mathcal{X}}$ and $\boldsymbol{\mathcal{X}}'$, respectively, are related through the relation

$$\mathbf{F}' = \mathbf{Q}(t)\mathbf{F}. \tag{15}$$

The arbitrariness of the rotation tensor $\mathbf{Q}(t)$ implies that it is possible to find infinitely many deformation gradients $\mathbf{F}$ generating the same right Cauchy–Green tensor. Therefore, even though the integrability condition (11) assures the existence of a rotation tensor, its uniqueness cannot be ensured. This indeterminacy can be resolved by assigning a value of $\boldsymbol{\mathcal{R}}$ at a particular point. Although one could argue that the question of uniqueness is not important here because the curl of all feasible deformation gradients generating a prescribed $\mathbf{C}$ must be zero, it is more reasonable to derive a condition on $\boldsymbol{\mathcal{U}}$ without introducing any potential indeterminacy. Even so, compatibility condition (12) is inarguably valid for any preassigned deformation gradient. Implications of this condition are discussed in Sec. 4.

3.2 Derivation of the compatibility condition

We now derive a compatibility condition for $\boldsymbol{\mathcal{U}}$ starting from Riemann's theorem. It has already been stated that the necessary and sufficient condition for existence and uniqueness of a finite deformation for a preassigned Cauchy–Green tensor is a vanishing of the Riemann curvature tensor [1,3,20,24]. This condition is valid irrespective of which decomposition of the deformation gradient is used, since **C** serves as the metric for any Lagrangian frame of reference. Cholesky factorization of **C** ensures the existence of a *unique* $\boldsymbol{\mathcal{U}}$ for a given **C** (Eq. 5). Therefore, we are interested in finding the restriction imposed on $\boldsymbol{\mathcal{U}}$ caused by a vanishing of the Riemann curvature tensor. Thus, in this approach, we consider **C** as the primary variable and plausibly avoid any issue for indeterminacy of a deformation gradient.

Differentiation of Eq. (4) immediately leads to

$$C_{ij,k} = \mathcal{U}_{mi,k}\mathcal{U}_{mj} + \mathcal{U}_{mi}\mathcal{U}_{mj,k}, \tag{16}$$

Writing $\mathtt{G}$ in terms of $\boldsymbol{\mathcal{U}}$ using Eq. (16) and substituting in Eq. (7), we get

$$\begin{aligned}\mathbb{R}_{ijkl} = {} & \frac{1}{2}[-\mathtt{W}_{mkl,j}\mathcal{U}_{mi} + \mathtt{W}_{mkl,i}\mathcal{U}_{mj} - \mathtt{W}_{mij,l}\mathcal{U}_{mk} + \mathtt{W}_{mij,k}\mathcal{U}_{ml} - \mathtt{W}_{mij}\mathtt{W}_{mkl}] + \frac{1}{4}[\mathtt{W}_{mlj}\mathtt{W}_{mik} \\ & + \mathtt{W}_{mil}\mathtt{W}_{mjk} + (\mathtt{W}_{nql}\mathtt{D}_{rkj} + \mathtt{W}_{nkq}\mathtt{D}_{rlj})\mathcal{U}^{-1}_{qr}\mathcal{U}_{ni} + \mathtt{W}_{nqi}\mathtt{D}_{rkj}\mathcal{U}_{nl}\mathcal{U}^{-1}_{qr} + \mathtt{W}_{mpj}\mathtt{D}_{ril}\mathcal{U}_{mk}\mathcal{U}^{-1}_{pr} \\ & + (\mathtt{D}_{ril}\mathtt{W}_{mpk} + \mathtt{D}_{rik}\mathtt{W}_{mlp})\mathcal{U}_{mj}\mathcal{U}^{-1}_{pr} + \mathtt{D}_{rlj}\mathtt{W}_{niq}\mathcal{U}_{nk}\mathcal{U}^{-1}_{qr} + \mathtt{D}_{rik}\mathtt{W}_{mjp}\mathcal{U}_{ml}\mathcal{U}^{-1}_{pr} \\ & + \mathtt{W}_{nql}\mathtt{W}_{mpj}\mathcal{U}_{mk}\mathcal{U}_{ni}\mathcal{U}^{-1}_{pr}\mathcal{U}^{-1}_{qr} + \mathtt{W}_{nqi}\mathtt{W}_{mpj}\mathcal{U}_{mk}\mathcal{U}_{nl}\mathcal{U}^{-1}_{pr}\mathcal{U}^{-1}_{qr} \\ & + (\mathtt{W}_{nql}\mathtt{W}_{mpk} + \mathtt{W}_{nqk}\mathtt{W}_{mlp})\mathcal{U}_{mj}\mathcal{U}_{ni}\mathcal{U}^{-1}_{pr}\mathcal{U}^{-1}_{qr} + \mathtt{W}_{niq}\mathtt{W}_{mkp}\mathcal{U}_{mj}\mathcal{U}_{nl}\mathcal{U}^{-1}_{pr}\mathcal{U}^{-1}_{qr} \\ & + \mathtt{W}_{nkq}\mathtt{W}_{mpj}\mathcal{U}_{ml}\mathcal{U}_{ni}\mathcal{U}^{-1}_{pr}\mathcal{U}^{-1}_{qr} + \mathtt{W}_{niq}\mathtt{W}_{mpj}\mathcal{U}_{ml}\mathcal{U}_{nk}\mathcal{U}^{-1}_{pr}\mathcal{U}^{-1}_{qr} + \mathtt{W}_{niq}\mathtt{W}_{mpl}\mathcal{U}_{mj}\mathcal{U}_{nk}\mathcal{U}^{-1}_{pr}\mathcal{U}^{-1}_{qr}],\end{aligned} \tag{17}$$

where $\mathtt{D}_{abc} = \mathcal{U}_{ab,c} + \mathcal{U}_{ac,b}$ and $\mathtt{W}_{abc} = \mathcal{U}_{ab,c} - \mathcal{U}_{ac,b}$ are third-order tensors, symmetric and skew-symmetric in b and c, respectively.

It is convenient to write Eq. (7) in tensor notation for algebraic manipulation. Transpositions of fourth-order tensors used in Kintzel and Başar [12] are particularly helpful in this case. However, transpositions for third-order tensors were not provided. Therefore, we define these transpositions in a way that they are consistent with Kintzel and Başar's transpositions for fourth-order tensors. A list of transpositions for second-, third- and fourth-order tensors is provided in Appendix A. Next, we write Eq. (7) in tensor notation:

$$\begin{aligned}\mathbb{R} = {} & \frac{1}{2}[-\mathbb{R}_1 + \mathbb{R}_1^{dl} + [\mathbb{R}_1^{D}]^{dr} - \mathbb{R}_1^{D}] + \frac{1}{4}[-2\mathbb{R}_2 + [\mathbb{R}_2^{dr}]^{ti} + [\mathbb{R}_2^{ti}]^{dr}] \\ & + \frac{1}{4}[[\mathbb{R}_3^{dr}]^{ti} + [\mathbb{R}_3^{dr}]^{to} + [\mathbb{R}_3^{ti}]^{T} + [\mathbb{R}_3^{to}]^{T} - [\mathbb{R}_3^{to}]^{dl} - [\mathbb{R}_3^{ti}]^{dl} - [[\mathbb{R}_3^{dr}]^{ti}]^{dr} - [[\mathbb{R}_3^{dr}]^{to}]^{dr}] \\ & + \frac{1}{4}[[\mathbb{R}_4^{ti}]^{dr} + [[\mathbb{R}_4^{dl}]^{ti}]^{dr} + [[\mathbb{R}_4^{dr}]^{ti}]^{dr} + [\mathbb{R}_4^{to}]^{dl} - [\mathbb{R}_4^{dr}]^{ti} - \mathbb{R}_4^{ti} - [\mathbb{R}_4^{dl}]^{ti} - [\mathbb{R}_4^{to}]^{T}],\end{aligned} \tag{18}$$

where $\mathbb{T}_{mjkl} = \mathtt{W}_{mkl,j}$ and

$$\mathbb{R}_1 = \boldsymbol{\mathcal{U}}^T\mathbb{T}; \quad \mathbb{R}_2 = \mathtt{W}^D\mathtt{W}; \quad \mathbb{R}_3 = \boldsymbol{\mathcal{U}}^T\mathtt{W}\boldsymbol{\mathcal{U}}^{-1}\mathtt{D}; \quad \mathbb{R}_4 = (\boldsymbol{\mathcal{U}}^T\mathtt{W}\boldsymbol{\mathcal{U}}^{-1})(\boldsymbol{\mathcal{U}}^T\mathtt{W}\boldsymbol{\mathcal{U}}^{-1})^t. \tag{19}$$

From the definitions for $\mathtt{W}$ and $\mathtt{D}$, it is easily understood that $\mathtt{W}^T = -\mathtt{W}$ and $\mathtt{D}^T = \mathtt{D}$. Now, it is important to investigate similar symmetries of the fourth-order tensors $\mathbb{T}$ and $\mathbb{R}_m$, $m = 1, 2, 3, 4$. From the definition of $\mathbb{T}$, it follows that $\mathbb{T}^{dr} = -\mathbb{T}$ because $\mathtt{W}^T = -\mathtt{W}$. Using the symmetries mentioned above, we obtain $\mathbb{R}_1^{dr} = -\mathbb{R}_1$, $\mathbb{R}_2^{dr} = -\mathbb{R}_2$ and $\mathbb{R}_3^{dr} = \mathbb{R}_3$. Symmetry of $\mathbb{R}_4$ is obtained from its definition, and it is $\mathbb{R}_4^t = \mathbb{R}_4$. Using these symmetries, we attain

$$\begin{aligned}\mathbb{R} = {} & \frac{1}{2}[(\mathbb{R}_1^{dl} - \mathbb{R}_1) + (\mathbb{R}_1^{dl} - \mathbb{R}_1)^D] + \frac{1}{4}[-2\mathbb{R}_2 + [\mathbb{R}_2^{dr}]^{ti} + [\mathbb{R}_2^{ti}]^{dr}] \\ & + \frac{1}{4}[(\mathbb{R}_3^{ti} + \mathbb{R}_3^{to}) + [(\mathbb{R}_3^{ti} + \mathbb{R}_3^{to})]^T - [(\mathbb{R}_3^{ti} + \mathbb{R}_3^{to})]^{dl} - [(\mathbb{R}_3^{ti} + \mathbb{R}_3^{to})]^{dr}] \\ & + \frac{1}{4}[(-\mathbb{R}_4^{ti} - [\mathbb{R}_4^{dr}]^{ti} + [\mathbb{R}_4^{ti}]^{dr} + [[\mathbb{R}_4^{dr}]^{ti}]^{dr}) + (-[\mathbb{R}_4^{ti}]^T - [\mathbb{R}_4^{dl}]^{ti} + [\mathbb{R}_4^{ti}]^{dl} + [[\mathbb{R}_4^{dl}]^{ti}]^{dr})].\end{aligned} \tag{20}$$

Other important symmetries come from the definition for the Riemann curvature tensor, viz., $\mathbb{R}^{dr} = -\mathbb{R}$ and $\mathbb{R}^{D} = \mathbb{R}$. These are ensured by the symmetries of $\mathbb{R}_1$, $\mathbb{R}_2$ and $\mathbb{R}_3$ and the interrelations of the transpositions (mentioned in Appendix B) when $\mathbb{R}$ is written in tensor notation.

We write Eq. (20) in matrix form in the Cartesian basis $\{\tilde{e}_I\}$ in order to use the upper-triangular property of $\boldsymbol{\mathcal{U}}$, which is essential in our derivation. We define the following functions to simplify our calculation:

$$\begin{aligned}
f_1(\mathbb{R}_1) &= f_1(\boldsymbol{\mathcal{U}}, \mathbb{T}) = (\mathbb{R}_1^{dl} - \mathbb{R}_1) + (\mathbb{R}_1^{dl} - \mathbb{R}_1)^D,\\
f_2(\mathbb{R}_2) &= f_2(\mathtt{W}) = -2\mathbb{R}_2 + [\mathbb{R}_2^{dr}]^{ti} + [\mathbb{R}_2^{ti}]^{dr},\\
f_3(\mathbb{R}_3) &= f_3(\boldsymbol{\mathcal{U}}, \mathtt{W}, \mathtt{D}) = (\mathbb{R}_3^{ti} + \mathbb{R}_3^{to}) + [(\mathbb{R}_3^{ti} + \mathbb{R}_3^{to})]^T - [(\mathbb{R}_3^{ti} + \mathbb{R}_3^{to})]^{dl} - [(\mathbb{R}_3^{ti} + \mathbb{R}_3^{to})]^{dr},\\
f_4(\mathbb{R}_4) &= f_4(\boldsymbol{\mathcal{U}}, \mathtt{D}, \mathtt{W}) = (-\mathbb{R}_4^{ti} - [\mathbb{R}_4^{dr}]^{ti} + [\mathbb{R}_4^{ti}]^{dr} + [[\mathbb{R}_4^{dr}]^{ti}]^{dr}) + (-[\mathbb{R}_4^{ti}]^T - [\mathbb{R}_4^{dl}]^{ti}.\\
&\quad + [\mathbb{R}_4^{ti}]^{dl} + [[\mathbb{R}_4^{dl}]^{ti}]^{dr})
\end{aligned} \tag{21}$$

The third-order tensors $\mathtt{W}$ and $\mathtt{D}$ take the following forms in $\tilde{e}_I$:

$$[\mathtt{W}_{ijk}] = \begin{bmatrix} 0 & W_1 & W_3 & -W_1 & 0 & W_6 & -W_3 & -W_6 & 0\\ 0 & W_2 & W_4 & -W_2 & 0 & W_7 & -W_4 & -W_7 & 0\\ 0 & 0 & W_5 & 0 & 0 & W_8 & -W_5 & -W_8 & 0 \end{bmatrix}, \tag{22}$$

where $W_1 = \gamma_{,1} - a_{,2}$; $W_2 = b_{,1}$; $W_3 = \beta_{,1} - a_{,3}$; $W_4 = \alpha_{,1}$; $W_5 = c_{,1}$; $W_6 = \beta_{,2} - \gamma_{,3}$; $W_7 = \alpha_{,2} - b_{,3}$ and $W_8 = c_{,2}$, and

$$[\mathtt{D}_{ijk}] = \begin{bmatrix} 2a_{,1} & \gamma_{,1} + a_{,2} & \beta_{,1} + a_{,3} & a_{,2} + \gamma_{,1} & 2\gamma_{,2} & \beta_{,2} + \gamma_{,3} & \gamma_{,3} + \beta_{,1} & \gamma_{,3} + \beta_{,2} & 2\beta_{,3}\\ 0 & b_{,1} & \alpha_{,1} & b_{,1} & 2b_{,2} & \alpha_{,2} + b_{,3} & \alpha_{,1} & b_{,3} + \alpha_{,2} & 2\alpha_{,3}\\ 0 & 0 & c_{,1} & 0 & 0 & c_{,2} & c_{,1} & c_{,2} & 2c_{,3} \end{bmatrix}. \tag{23}$$

Next, we write the functions in Eq. (21) in the basis $\{\tilde{e}_I\}$. When written in matrix form, each of these functions forms a 9×9 square matrix consisting of nine 3×3 block matrices. Some important features of these matrices are listed below.

- The diagonal elements of the matrix $f_1(\mathbb{R}_1)$ are zero.
- $f_m(\mathbb{R}_m)^D = f_m(\mathbb{R}_m)$, i.e., $[f_m(\mathbb{R}_m)]_{klij} = [f_m(\mathbb{R}_m)]_{ijkl}$.
- The diagonal blocks of matrices $f_n(\mathbb{R}_n)$, $n = 2, 3, 4$, are zero.
- The sub-matrices forming the off-diagonal blocks of $f_n(\mathbb{R}_n)$ are skew-symmetric, i.e., $[f_n(\mathbb{R}_n)]_{ijkl} = -[f_n(\mathbb{R}_n)]_{jikl}$.
- The diagonal elements of each off-diagonal block of $f_n(\mathbb{R}_n)$ are zero.
- The off-diagonal blocks of $f_n(\mathbb{R}_n)$ are skew-symmetric, i.e., $[f_n(\mathbb{R}_n)]_{ijkl} = -[f_n(\mathbb{R}_n)]_{ijlk}$.

Note that even if the symmetries of $\mathbb{R}$ ($\mathbb{R}^{dr} = -\mathbb{R}$ and $\mathbb{R}^D = \mathbb{R}$) are easily realized in tensor notation, they are not obvious from the structure of $\mathbb{R}_m$. Two important observations are made based upon the structure of $f_m(\mathbb{R}_m)$ and their associated symmetry properties.

- Since the diagonal blocks of $f_n(\mathbb{R}_n)$ are zero and $\mathbb{R}_{ijkl} = \sum_{m=1}^{4} f_m(\mathbb{R}_m)$, the only remaining nonzero elements in the diagonal blocks of $\mathbb{R}$ arise from $f_1(\mathbb{R}_1)$. Consequently, elements in the diagonal blocks of $f_1(\mathbb{R}_1)$ must be equal to zero.
- The off-diagonal blocks of $f_n(\mathbb{R}_n)$ are skew-symmetric, whereas this is not the case for $f_1(\mathbb{R}_1)$. The off-diagonal blocks of $f_1(\mathbb{R}_1)$ must be either skew-symmetric or zero because the $f_m(\mathbb{R}_m)$ add up to the Riemann curvature tensor whose off-diagonal blocks are skew-symmetric, i.e., $\mathbb{R}^{dr} = -\mathbb{R}$. For generality, the off-diagonal blocks of $f_1(\mathbb{R}_1)$ are taken to be skew-symmetric.

These two restrictions on $f_1(\mathbb{R}_1)$ give rise to 18 equations, out of which only 15 are independent. These equations are made up of products between the fourth-order tensor $\mathbb{T}$, the Laplace stretch $\boldsymbol{\mathcal{U}}$ and its transpose. The symmetry $\mathbb{R}^D = \mathbb{R}$ is automatically preserved by the structure of $f_m(\mathbb{R}_m)$. The equations are provided in Appendix C.1.

Next, we focus on the off-diagonal elements in the off-diagonal blocks of the Riemann curvature tensor. Other entries in $\mathbb{R}$ are identically zero. Note that most of these elements are either equal to or the negative of some other elements due to the symmetry $\mathbb{R}^D = \mathbb{R}$. In fact, only six of these elements are independent.[5] These

[5] Note that the Ricci tensor, obtained by contracting a Riemann curvature tensor, has nine components. The symmetry arguments of a Riemann curvature tensor ensure that only six of these nine components are independent.

elements are $\mathbb{R}_{1212}$, $\mathbb{R}_{1312}$, $\mathbb{R}_{2312}$, $\mathbb{R}_{1313}$, $\mathbb{R}_{2313}$ and $\mathbb{R}_{2323}$. Thus, equating these elements to zero, we find six independent equations. These equations constitute multiple terms from $\mathbb{R}_m$.

The set of six equations holds for all possible deformations because the Riemann curvature tensor must be zero for any prescribed Cauchy–Green tensor to ensure a valid finite deformation. Since the system of equations is homogeneous, $\mathtt{W} = 0$ is a trivial solution. *One can show that the trivial solution is the only solution to the system of equations if all the components of Laplace stretch* $\boldsymbol{\mathcal{U}}$ *in Eq.* (1) *are deemed to be independent.* However, the trivial solution is too restrictive, because it holds only when the rotation tensor $\boldsymbol{\mathcal{R}}$ is homogeneous. This is due to an interdependence between some of the components of $\boldsymbol{\mathcal{U}}$.

Up to this point in this paper, we have adopted Srinivasa [22]'s physical interpretation for the components of Laplace stretch, i.e., Eq. (1), because it results in simpler expressions in our derivation. Freed et al. [8–10] adopted a different physical interpretation of

$$[\boldsymbol{\mathcal{U}}_{ij}] = \begin{bmatrix} a & a\tilde{\gamma} & a\tilde{\beta} \\ 0 & b & b\tilde{\alpha} \\ 0 & 0 & c \end{bmatrix} \tag{24}$$

wherein a, b, c have the same physical interpretations as their counterparts in Srinivasa's kinematic variables. In contrast, the shears $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$ in Eq. (24) denote *magnitudes* of shear, whereas their interpretations in Eq. (1) designate *extents* of shear. Clearly, all elements of Laplace stretch in Eq. (1) are not independent. Specifically, the extent of shears α, β and γ explicitly depends upon the elongations a or b through

$$\alpha = b\tilde{\alpha}, \quad \beta = a\tilde{\beta}, \quad \gamma = a\tilde{\gamma}. \tag{25}$$

Apart from couplings between a, γ, β and b, α, there is no reason for other shears or elongations to be coupled, e.g., a variation in the extent of shear γ is not expected to depend on a variation in elongation c. This is a consequence of the fact that the $\tilde{e}_1$ and $\tilde{e}_1 \times \tilde{e}_2$ coordinate directions remain invariant under transformation $\boldsymbol{\mathcal{U}}$ [16,17].

To determine restrictions on components of $\mathtt{W}$ (or on those of Laplace stretch $\boldsymbol{\mathcal{U}}$) imposed by compatibility, we pick specific conditions that make some of the components of $\mathtt{W}$ or their coefficients zero. One has to be careful in choosing these conditions to make sure that these conditions have no effect on the component of interest, and that the couplings between shear and elongation in Eq. (25) are taken into account. For example, one should not pick b to vary in a certain way to determine conditions on the variation of α and vice versa. Since the cases are chosen in a way that restrictions on other components of $\mathtt{W}$ do not have any effect on the component of interest, it is evident that the condition on the component of interest emerging from a vanishing of Riemann curvature tensor is valid for all feasible deformations.

It is important to note that some elements of $f_1(\mathbb{R}_1)$ become zero in order to preserve the restrictions on it as stated earlier. Because couplings between shears and elongations, apart from the ones between a, β, γ and b, α are not expected, one can show that the terms involving derivatives of W_p, $p = 1, \ldots, 8$, do not appear in the system of six equations arising from a vanishing the of Riemann curvature tensor. See Appendix C.1 for details.

$\mathbb{R}_{1212} = 0$ leads to:[6]

$$\begin{aligned} &2a\cancel{W_{1,2}} + \left(-\frac{\beta^2}{c^2} + \frac{4\alpha\gamma\beta}{bc^2} - \frac{4\alpha^2\gamma^2}{b^2c^2}\right)W_1^2 - \frac{\alpha^2}{c^2}W_2^2 - \frac{\gamma^2}{c^2}W_3^2 - \frac{b^2}{c^2}W_4^2 - \frac{a^2}{c^2}W_6^2 + \frac{4ab_{,2}}{b}W_1 \\ &- \frac{4a_{,1}b}{a}W_2 + \left(\frac{2\alpha\beta}{c^2} - \frac{4\gamma}{b} - \frac{4a\alpha^2}{bc^2}\right)W_1W_2 + \left(\frac{4\gamma^2\alpha}{bc^2} - \frac{2\gamma\beta}{c^2}\right)W_1W_3 + \frac{2b\beta}{c^2}W_1W_4 \\ &+ \left(\frac{2a\beta}{c^2} - \frac{4a\alpha\gamma}{bc^2}\right)W_1W_6 - \frac{4a\alpha}{c^2}W_1W_7 + \left(\frac{6\alpha\gamma}{c^2} - \frac{4b\beta}{c^2}\right)W_2W_3 + \frac{2a\alpha}{c^2}W_2W_6 + \frac{2b\alpha}{c^2}W_2W_4 \\ &- \frac{2\gamma b}{c^2}W_3W_4 + \frac{2a\gamma}{c^2}W_3W_6 + \frac{4ab}{c^2}W_3W_7 - \frac{2ab}{c^2}W_4W_6 = 0. \end{aligned} \tag{26}$$

Now, we pick specific cases and determine the conditions that must be met to satisfy Eq. (26). First, we consider a deformation where a, γ and β are arbitrary constants, i.e., W_1, W_3 and W_6 are zero. Equation (26) reduces to

$$\alpha W_2 - bW_4 = 0. \tag{27}$$

[6] The term $W_{1,2}$ becomes zero according to Eq. (C.1.19).

Note that W_2 and W_4 involve derivatives of b and α, respectively. Because variations of b and α are not expected to depend upon variations of any other components of Laplace stretch, we conclude that the condition in Eq. (27) is valid irrespective of chosen conditions on a, β and γ, and thus, for all feasible deformations. Equation (27) further implies that

$$\tilde{\alpha}_{,1} = 0. \tag{28}$$

It is evident from Eq. (27) that if b is a function of X_1, then α too must be a function of X_1, and vice versa. This is corroborated by the relation $\alpha_{,1} = \tilde{\alpha} b_{,1}$ obtained by differentiating $\alpha = \tilde{\alpha} b$ and using Eq. (28).

Next, we allow a, β and γ to vary arbitrarily and choose b and α to be arbitrary constants. In this case, Eq. (26) reduces to

$$\beta W_1 + \gamma W_3 - a W_6 = 0. \tag{29}$$

By a similar argument, we conclude that this condition has to be satisfied for all possible deformations because, apart from their interdependence a, β and γ, they do not depend on the other components of $\boldsymbol{\mathcal{U}}$. Expanding in terms of $\tilde{\alpha}$, $\tilde{\beta}$, Eq. (29) yields:

$$(a^2 \tilde{\beta} \tilde{\gamma})_{,1} - (a^2 \tilde{\beta})_{,2} - a \tilde{\gamma} a_{,3} + a^2 \tilde{\gamma}_{,3} = 0. \tag{30}$$

Nothing else can be concluded about the other components of $\mathtt{W}$ from this equation, because the components W_5, W_7, W_8 either appear in Eq. (26) as a product with another component of $\mathtt{W}$, e.g., terms like $-\dfrac{4a\alpha}{c^2} W_1 W_7$, or they do not appear at all. To find conditions on a component of $\mathtt{W}$, we need at least one stand-alone term associated with the component of interest that does not appear as a product with other components.

For conditions on W_5, we appeal to the equation arising from equating $\mathbb{R}_{1313}$ to zero. Similar conditions on W_7 and W_8 are derived from the equation $\mathbb{R}_{2323} = 0$. These two equations are provided in Appendix "C.2." By choosing suitable conditions on components of $\boldsymbol{\mathcal{U}}$, one can show that W_5 must be zero in order to satisfy equations (C.2.1), because all other terms are identically zero whenever Eqs. (27) and (29) are employed, thus,

$$W_5 = 0 \implies c_{,1} = 0. \tag{31}$$

We use Eq. (C.2.2) to find conditions on W_7 and W_8. Choosing a to be an arbitrary constant and $\tilde{\beta} = \tilde{\gamma} = 0$ (or b, α to be arbitrary constants) and using the conditions derived above and independence of b, α (or a, β, γ) and c, one can conclude that

$$W_7 = 0 \implies (b\tilde{\alpha})_{,2} = b_{,3} \tag{32}$$

and finally,

$$W_8 = 0 \implies c_{,2} = 0. \tag{33}$$

The fact that $c_{,1} = 0$ and $c_{,2} = 0$, where c is the elongation normal to the 12 plane, is consistent with the fact that the normal to the 12 plane is invariant under transformations of the Laplace stretch [17].

All other relevant elements of the Riemann curvature tensor, viz., $\mathbb{R}_{1312}$, $\mathbb{R}_{2312}$ and $\mathbb{R}_{2313}$, become identically zero whenever the conditions derived above are employed. Therefore, a vanishing Riemann curvature tensor implies that Eqs. (27)–(33) must be satisfied. On the other hand, for any feasible deformation that satisfies these equations, the Riemann curvature tensor vanishes. Furthermore, in view of the coupling between certain shears and elongations mentioned earlier, it is not possible to find a deformation for which these equations are not satisfied, yet the Riemann curvature tensor goes to zero. Therefore, these equations serve as the necessary and sufficient conditions for vanishing of a Riemann curvature tensor and, hence, for a deformation to be compatible.

4 Discussion

4.1 Compatibility condition of Laplace stretch for prescribed **F**

Whenever a deformation gradient is prescribed, the necessary and sufficient integrability condition for the existence of a unique deformation map is curl(**F**) $= \mathbf{0}$, which implies

$$F_{ij,k} = F_{ik,j}. \tag{34}$$

Using the definition for the Christoffel symbol in Eq. (6), a relationship between **C**, **F** and Eq. (34) leads to

$$F_{pi} F_{pj,k} = \mathsf{G}_{jki} \implies F_{ij,k} = F_{ip} C^{-1}_{pq} \mathsf{G}_{jkq}. \tag{35}$$

Differentiating Eq. (35) with respect to $\boldsymbol{X}$ and eliminating the first derivative of **F** by using the later part of Eq. (35), we get

$$F_{pi} F_{pj,kl} = \mathsf{G}_{jki,l} - C^{-1}_{pq} \mathsf{G}_{jkp} \mathsf{G}_{ilq}. \tag{36}$$

Since $F_{pi} F_{pj,kl} = F_{pi} F_{pj,lk}$, one finally gets $\mathbb{R}_{ijkl} = 0$. Therefore, Eqs. (27)–(33) must be satisfied. Thus, the only restriction of $\boldsymbol{\mathcal{U}}$ provided by the integrability condition of a prescribed deformation gradient is the same as the restriction pertaining to a given Cauchy–Green tensor (or Laplace stretch). In fact, curl(**F**) $= \mathbf{0}$ does not have any direct effect on $\boldsymbol{\mathcal{U}}$. This can be shown from a decomposition of deformation gradient into an orthogonal rotation and the Laplace stretch.

$$\mathrm{curl}(\mathbf{F}) = \mathbf{0} \implies (\mathcal{R}\mathcal{U})_{ij,k} = (\mathcal{R}\mathcal{U})_{ik,j}. \tag{37}$$

Expansion of this expression, and use of Eq. (9) yield

$$\mathsf{Z}_{pnk} \mathcal{U}_{nj} - \mathsf{Z}_{pnj} \mathcal{U}_{nk} = \mathcal{U}_{pk,j} - \mathcal{U}_{pj,k}. \tag{38}$$

Substitution of Eq. (10) into Eq. (38) implies $\mathsf{G}_{ijk} = \mathsf{G}_{ikj}$, which immediately follows from the definition of the Christoffel symbol. Note that Lembo's compatibility condition is an alternative statement for Eq. (38).

4.2 Implication of compatibility condition for Laplace stretch and utility of **QR** factorization

The derived compatibility conditions for Laplace stretch restrict the dependence of its components on spatial variables X_I. In view of physical interpretations of these components, one can easily understand the dependence of elongation and shear, i.e., deformations in all six degrees of freedom on the spatial variables of a reference configuration and their interdependence to generate a valid finite deformation field.

As discussed before, coupling between certain shears and elongations plays a crucial role in deriving these conditions. If all the components of Laplace stretch are assumed to be independent, then one can show that the trivial solution curl($\boldsymbol{\mathcal{U}}$) $= \mathbf{0}$ is the only solution for the system of equations, which further implies that the rotation tensor has to be homogeneous. When all the components of Laplace stretch are considered to be dependent on each other, the necessary and sufficient compatibility condition is a vanishing of Riemann curvature tensor $\mathbb{R}$. Note that these conditions are the same as the ones obtained by equating the six independent elements of $\mathbb{R}$, expressed in terms of elements of **C**, to zero. Although these conditions ensure that the reference configuration is a Euclidean space for which the right Cauchy–Green tensor acts as a metric, they lack any direct physical interpretation. Clearly, an inability to determine physical meaning for the components of **C** is responsible for this shortcoming. In fact, it is not possible to understand the interdependence between the components of **C** whenever a traditional polar decomposition of the deformation gradient is used and, hence, all the components are deemed to be coupled.

However, with the use of a **QR** decomposition, it is easy to understand that not all components of the Laplace stretch are coupled. Specifically, the only existing couplings are (i) a, γ and β; (ii) b and α. The elongation c does not depend on any of the components of Laplace stretch. The decoupling of certain components of Laplace stretch demands that five constituents from the Riemann curvature tensor, expressed in terms of elements of Laplace stretch and their derivatives, must be individually zero to ensure compatibility. Thus, the derived compatibility conditions are vastly different from the set of six equations arising from a vanishing of Riemann curvature tensor. Obviously, if these constituents are zero, the Riemann curvature tensor vanishes.

When these constituents are not individually zero, one can pick a suitable condition on some components of Laplace stretch such that the other terms are not coupled with the former ones and show that Riemann curvature tensor does not identically go to zero. For example, one can pick a, β, γ, c to be arbitrary constants and show that the equations arising from a vanishing of Riemann curvature tensor leave residues of the form $p(\alpha b_{,1} - b\alpha_{,1})^m + q(\alpha_{,2} - b_{,3})^n$ where p, q are arbitrary constants and m, n are integers. Thus, the equations are not identically satisfied. Therefore, when some of the elements of Laplace stretch are not coupled, the derived equations (27)–(33) must be satisfied in order for the Riemann curvature tensor to vanish. Thus, they serve as the necessary and sufficient conditions for existence and uniqueness of a valid finite deformation field for a prescribed Cauchy–Green tensor or for a prescribed Laplace stretch.

5 Summary

In this paper, the compatibility conditions for Laplace stretch arising from a Gram–Schmidt factorization of the deformation gradient have been obtained. We show that these conditions must be satisfied to ensure existence and uniqueness of a deformation map for prescribed $\mathbf{C}$ or $\boldsymbol{\mathcal{U}}$ when coupling between certain shears and elongations is in action. When off-diagonal terms of Laplace stretch $\boldsymbol{\mathcal{U}}$ are expressed in terms of extents of shear α, β, γ according to Srinivasa [22]'s notation, the compatibility conditions are

$$\begin{aligned}
&\alpha b_{,1} - b\alpha_{,1} = 0, \\
&\beta\left(\gamma_{,1} - a_{,2}\right) + \gamma\left(\beta_{,1} - a_{,3}\right) - a\left(\beta_{,2} - \gamma_{,3}\right) = 0, \\
&c_{,1} = 0, \\
&\alpha_{,2} - b_{,3} = 0, \\
&c_{,2} = 0.
\end{aligned} \tag{39}$$

When expressed in terms of magnitudes of shear $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$ per the notation of Freed [9], the structures of the compatibility conditions slightly change. In terms of $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$, the compatibility conditions can be expressed as

$$\begin{aligned}
&\tilde{\alpha}_{,1} = 0, \\
&(a^2\tilde{\beta}\tilde{\gamma})_{,1} - (a^2\tilde{\beta})_{,2} - a\tilde{\gamma}a_{,3} + a^2\tilde{\gamma}_{,3} = 0, \\
&c_{,1} = 0, \\
&(b\tilde{\alpha})_{,2} - b_{,3} = 0, \\
&c_{,2} = 0.
\end{aligned} \tag{40}$$

Restriction on $\boldsymbol{\mathcal{U}}$ imposed by the integrability condition of a given deformation gradient has also been explored and shown to be the same as that for a prescribed Cauchy–Green tensor or a prescribed Laplace stretch. Coupling between certain components of Laplace stretch representing shear and elongation plays a crucial role in deriving these conditions. Finally, the implication of these compatibility conditions and the utility of Gram–Schmidt factorization of deformation gradient in this context is discussed.

Acknowledgements The authors thank Dr. John D. Clayton for his comments.

A Transpositions of tensors

Fourth-order tensor:

$$\begin{aligned}
\mathbb{A}^T &= (\mathbb{A}^{ijkl} e_i \otimes e_j \otimes e_k \otimes e_l)^T = \mathbb{A}^{ijkl} e_j \otimes e_i \otimes e_l \otimes e_k = \mathbb{A}^{jilk} e_i \otimes e_j \otimes e_k \otimes e_l \\
\mathbb{A}^{ti} &= (\mathbb{A}^{ijkl} e_i \otimes e_k \otimes e_j \otimes e_l)^{ti} = \mathbb{A}^{ikjl} e_j \otimes e_i \otimes e_l \otimes e_k = \mathbb{A}^{ikjl} e_i \otimes e_j \otimes e_k \otimes e_l \\
\mathbb{A}^{to} &= (\mathbb{A}^{ijkl} e_i \otimes e_j \otimes e_k \otimes e_l)^{to} = \mathbb{A}^{ijkl} e_l \otimes e_j \otimes e_k \otimes e_i = \mathbb{A}^{ljki} e_i \otimes e_j \otimes e_k \otimes e_l \\
\mathbb{A}^{t} &= (\mathbb{A}^{ijkl} e_i \otimes e_j \otimes e_k \otimes e_l)^{t} = \mathbb{A}^{ijkl} e_l \otimes e_k \otimes e_j \otimes e_i = \mathbb{A}^{lkji} e_i \otimes e_j \otimes e_k \otimes e_l
\end{aligned}$$

$$\begin{aligned}
\mathbb{A}^D &= (\mathbb{A}^{ijkl} e_i \otimes e_j \otimes e_k \otimes e_l)^D = \mathbb{A}^{ijkl} e_k \otimes e_l \otimes e_i \otimes e_j = \mathbb{A}^{klij} e_i \otimes e_j \otimes e_k \otimes e_l \\
\mathbb{A}^{dl} &= (\mathbb{A}^{ijkl} e_i \otimes e_j \otimes e_k \otimes e_l)^{dl} = \mathbb{A}^{ijkl} e_j \otimes e_i \otimes e_k \otimes e_l = \mathbb{A}^{jikl} e_i \otimes e_j \otimes e_k \otimes e_l \\
\mathbb{A}^{dr} &= (\mathbb{A}^{ijkl} e_i \otimes e_j \otimes e_k \otimes e_l)^{dr} = \mathbb{A}^{ijkl} e_i \otimes e_j \otimes e_l \otimes e_k = \mathbb{A}^{ijlk} e_i \otimes e_j \otimes e_k \otimes e_l \\
\mathbb{A}^d &= (\mathbb{A}^{ijkl} e_i \otimes e_j \otimes e_k \otimes e_l)^d = \mathbb{A}^{ijkl} e_j \otimes e_i \otimes e_l \otimes e_k = \mathbb{A}^{jilk} e_i \otimes e_j \otimes e_k \otimes e_l
\end{aligned} \tag{A.1}$$

Third-order tensor:

$$\begin{aligned}
\mathtt{A}^T &= (\mathtt{A}^{ijk} e_i \otimes e_j \otimes e_k)^T = \mathtt{A}^{ijk} e_i \otimes e_k \otimes e_j = \mathtt{A}^{ikj} e_i \otimes e_j \otimes e_k \\
\mathtt{A}^t &= (\mathtt{A}^{ijk} e_i \otimes e_j \otimes e_k)^t = \mathtt{A}^{ijk} e_k \otimes e_j \otimes e_i = \mathtt{A}^{kji} e_i \otimes e_j \otimes e_k \\
\mathtt{A}^D &= (\mathtt{A}^{ijk} e_i \otimes e_j \otimes e_k)^D = \mathtt{A}^{ijk} e_j \otimes e_k \otimes e_i = \mathtt{A}^{jki} e_i \otimes e_j \otimes e_k
\end{aligned} \tag{A.2}$$

Second-order tensor:

$$\mathbf{A}^T = (\mathbf{A}^{ij} e_i \otimes e_j)^T = \mathbf{A}^{ij} e_j \otimes e_i = \mathbf{A}^{ji} e_i \otimes e_j \tag{A.3}$$

B Symmetries of $\mathbb{R}_m$

$\mathbb{R}_1$:

$$\begin{aligned}
&\mathbb{R}_1^{dr} = -\mathbb{R}_1; \quad [\mathbb{R}_1^{dl}]^{dr} = -\mathbb{R}_1^{dl}; \quad [[\mathbb{R}_1^D]^{dr}] = -\mathbb{R}_1^D; \\
&[[\mathbb{R}_1^D]^{dl}]^{dr} = -[\mathbb{R}_1^D]^{dl}; \quad [\mathbb{R}_1^D]^{dr} = [\mathbb{R}_1^{dl}]^D
\end{aligned} \tag{B.4}$$

$\mathbb{R}_2$:

$$\mathbb{R}_2^{dr} = -\mathbb{R}_2; \quad \mathbb{R}_2^D = \mathbb{R}_2; \quad \mathbb{R}_2^{ti} = [\mathbb{R}_2^{ti}]^D; \quad [\mathbb{R}_2^{ti}]^{dr} = [[\mathbb{R}_2^{ti}]^{dr}]^D \tag{B.5}$$

$\mathbb{R}_3$:

$$\begin{aligned}
&\mathbb{R}_3^{dr} = \mathbb{R}_3; \quad [\mathbb{R}_3^{to}]^T = [\mathbb{R}_3^{dl}]^{ti}; \quad [\mathbb{R}_3^{ti}]^T = [\mathbb{R}_3^{dl}]^{to}; \\
&[[\mathbb{R}_3^{ti}]^T]^{dr} = [\mathbb{R}_3^{ti}]^{dl}; \quad [[\mathbb{R}_3^{to}]^{dl}]^{dr} = [\mathbb{R}_3^{to}]^T; \quad [[\mathbb{R}_3^{to}]^{dl}]^D = [\mathbb{R}_3^{ti}]^{dl}; \quad [[\mathbb{R}_3^{ti}]^{dl}]^D = [\mathbb{R}_3^{to}]^{dl}; \\
&[[\mathbb{R}_3^{ti}]^{dr}]^D = [\mathbb{R}_3^{to}]^{dr} \quad [[\mathbb{R}_3^{to}]^{dr}]^D = [\mathbb{R}_3^{ti}]^{dr}; \quad [[\mathbb{R}_3^{to}]^T]^D = [\mathbb{R}_3^{dr}]^{ti}; \quad [[\mathbb{R}_3^{ti}]^T]^D = [\mathbb{R}_3^{dr}]^{to};
\end{aligned} \tag{B.6}$$

$\mathbb{R}_4$:

$$\begin{aligned}
&\mathbb{R}_4^t = \mathbb{R}_4; \quad [[\mathbb{R}_4^{to}]^T]^{dr} = [\mathbb{R}_4^{to}]^{dl}; \quad [[[\mathbb{R}_4^{dr}]^{ti}]^{dr}]^D = [[\mathbb{R}_4^{dl}]^{ti}]^{dr}; \\
&[[\mathbb{R}_4^{to}]^{dl}]^D = [\mathbb{R}_4^{to}]^{dl}; \quad [[\mathbb{R}_4^{dr}]^{ti}]^D = [\mathbb{R}_4^{dl}]^{ti}; \quad [[\mathbb{R}_4^{ti}]^{dr}]^D = [\mathbb{R}_4^{ti}]^{dr}; \quad [\mathbb{R}_4^{ti}]^D = [\mathbb{R}_4^{to}]^T
\end{aligned} \tag{B.7}$$

C System of equations

C.1 Equations arising from symmetries of $f_1(\mathbb{R}_1)$

Because the diagonal blocks of $f_1(\mathbb{R})$ are zero, we obtain:

$$
\begin{aligned}
&a W_{1,1} = 0, && \text{(C.1.1)}\\
&a W_{3,1} = 0, && \text{(C.1.2)}\\
&\beta W_{1,1} + \gamma W_{2,1} = \alpha W_{3,1} + b W_{4,1}, && \text{(C.1.3)}\\
&\alpha W_{1,2} + b W_{2,2} = 0, && \text{(C.1.4)}\\
&\alpha W_{6,2} + b W_{7,2} = 0, && \text{(C.1.5)}\\
&\beta W_{1,2} = -a W_{6,2}, && \text{(C.1.6)}\\
&\beta W_{6,3} + \gamma W_{7,3} + c W_{8,3} = 0, && \text{(C.1.7)}\\
&\beta W_{3,3} + \gamma W_{4,3} + c W_{5,3} = 0, && \text{(C.1.8)}\\
&\alpha W_{3,3} + b W_{4,3} = a W_{6,3}. && \text{(C.1.9)}
\end{aligned}
$$

From skew-symmetry of $f_1(\mathbb{R}_1)$, we get

$$
\begin{aligned}
&a W_{1,2} = -\gamma W_{1,1} - b W_{2,1}, && \text{(C.1.10)}\\
&a W_{3,2} = -\beta W_{1,1} - \alpha W_{2,1}, && \text{(C.1.11)}\\
&\gamma W_{6,1} + b W_{7,1} = \alpha W_{1,2} + \alpha W_{2,2} - \gamma W_{3,2} - b W_{4,2}, && \text{(C.1.12)}\\
&a W_{1,3} = a W_{6,1} - \gamma W_{1,3} - b W_{4,1}, && \text{(C.1.13)}\\
&a W_{3,3} = -\beta W_{3,1} - \alpha W_{4,1} - c W_{5,1}, && \text{(C.1.14)}\\
&\gamma W_{3,3} + b W_{4,3} = \beta W_{1,3} + \alpha W_{2,3} - \beta W_{6,1} - \alpha W_{7,1} - c W_{8,1}, && \text{(C.1.15)}\\
&a W_{6,2} = \gamma W_{2,3} + b W_{2,3} + \gamma W_{3,2} + b W_{4,2}, && \text{(C.1.16)}\\
&a W_{6,3} = -\beta W_{1,3} - \alpha W_{2,3} - \beta W_{3,2} - \alpha W_{4,2} - c W_{5,2}, && \text{(C.1.17)}\\
&\gamma W_{6,3} + b W_{7,3} = -\beta W_{6,2} - \alpha W_{7,2} - c W_{8,2}. && \text{(C.1.18)}
\end{aligned}
$$

Because a, γ, β and b, α are the only coupled elements of $\boldsymbol{\mathcal{U}}$, we conclude that

$$
\begin{aligned}
W_{1,1} &= W_{2,1} = W_{1,2} = W_{3,3} = W_{4,3} = W_{3,1} = W_{6,2} = W_{6,3} = W_{7,3} = W_{2,2} = W_{5,3} = W_{7,2}\\
&= W_{8,3} = W_{4,1} = W_{5,1} = W_{8,2} = W_{3,2} + W_{6,1} = W_{4,2} + W_{7,1} = W_{1,3} - W_{6,1} = W_{8,1} \qquad \text{(C.1.19)}\\
&= W_{5,2} = W_{2,3} - W_{7,1} = 0.
\end{aligned}
$$

Thus, no term involving derivatives of W_p, $p = 1, \ldots, 8$, appears in the 6 equations arising from equating off-diagonal elements of $\mathbb{R}$ to zero.

C.2 Equations arising from vanishing of Riemann curvature tensor

$\mathbb{R}_{1313} = 0$ leads to:

$$
\begin{aligned}
&a \cancel{W_{3,3}} + \beta \cancel{W_{3,1}} - \left(\frac{\beta^2}{b^2} + \frac{\beta^2\alpha^2}{b^2c^2}\right) W_1^2\\
&- \frac{2\alpha^4}{b^2c^2} W_2^2 + \left(2 + \frac{4a\alpha\gamma\beta}{b^2c^2} - \frac{4a\beta^2}{bc^2} - \frac{\alpha^2}{b^2} - \frac{\gamma^2}{b^2} - \frac{\alpha^2\gamma^2}{b^2c^2}\right) W_3^2\\
&+ 2W_5^2 - \left(\frac{\alpha^2\gamma^2}{b^2c^2} + \frac{a^2}{b^2}\right) W_6^2\\
&- \frac{2\beta\alpha^3}{b^2c^2} W_1 W_2 + \left(\frac{2\gamma\beta}{b^2} - \frac{2\beta\alpha}{b^2} - \frac{2\gamma\beta\alpha^2}{b^2c^2} + \frac{4\beta^2\alpha}{bc^2}\right) W_1 W_3 - \frac{4a\alpha}{b^2} W_1 W_7\\
&+ \left(-\frac{2\beta}{b} + \frac{4\alpha\gamma}{b^2} - \frac{4\gamma\alpha^3}{b^2c^2} + \frac{6\beta\alpha^2}{bc^2}\right) W_1 W_4 + \left(\frac{8\beta\alpha}{bc} + \frac{4\gamma c}{b^2} - \frac{4\gamma\alpha^2}{b^2 c}\right)\\
&+ \left(\frac{2\gamma\beta\alpha^2}{b^2c^2} - \frac{4a\beta}{b^2}\right) W_1 W_6\\
&- \frac{4ac}{b^2} W_1 W_8 + \left(\frac{2\gamma\alpha^3}{b^2c^2} - \frac{4\gamma\alpha}{b^2}\right) W_2 W_3 + \left(\frac{2\alpha^3}{bc^2} - \frac{2\alpha}{b}\right) W_2 W_4
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{4\alpha^2}{bc}W_2W_5+\left(\frac{2\gamma\alpha^2}{bc^2}-2\gamma\right)W_3W_4\\
&+\left(\frac{2a\alpha}{b^2}+\frac{2a\alpha^3}{b^2c^2}\right)W_2W_6+\frac{2\beta}{c}W_3W_5\\
&+\left(\frac{2\alpha\beta\gamma}{bc^2}+\frac{4a\beta\alpha}{bc^2}+\frac{4a\gamma}{b^2}-\frac{4a^2\alpha^2}{b^2c^2}\right)W_3W_6+\frac{4a\alpha^2}{bc^2}W_3W_7\\
&+\frac{4a\alpha}{bc}W_3W_8+\frac{2\alpha}{c}W_4W_5-\frac{2a\alpha^2}{bc^2}W_4W_6\\
&+\left(\frac{2a}{b}-\frac{2a\alpha}{bc}\right)W_5W_6+\left(\frac{4a\alpha_{,3}}{b}-\frac{4a\alpha c_{,3}}{bc}\right)W_1\\
&-\frac{4\alpha a_{,1}}{a}W_4+\left(\frac{4ac_{,3}}{c}-\frac{4\beta a_{,1}}{a}\right)W_3\\
&-\frac{4ca_{,1}}{a}W_5+\left(3-\frac{\alpha^2}{c^2}\right)W_4^2=0.
\end{aligned}
\tag{C.2.1}
$$

$\mathbb{R}_{2323}=0$ yields:

$$
\begin{aligned}
&\alpha\cancel{W_{6,3}}+b\cancel{W_{7,3}}+\left(\frac{\beta^2}{a^2}+\frac{\beta^2\xi^2}{a^2b^2c^2}\right)W_1^2-\left(\frac{\alpha^2}{a^2}+\frac{\alpha^2\xi^2}{a^2b^2c^2}\right)W_2^2-\frac{2\beta^2\xi}{abc^2}W_3^2\\
&-\left(\frac{b^2}{a^2}+\frac{2\alpha^2\xi}{abc}\right)W_4^2-\frac{\alpha\xi}{abc}W_5^2\\
&+\left(3-\frac{3\alpha^2}{c^2}\right)W_7^2+\left(3-\frac{2\alpha\beta\gamma}{bc^2}-\frac{\beta^2}{c^2}\right)W_6^2-\left(\frac{2\beta\alpha}{a^2}+\frac{2\beta\alpha\xi^2}{a^2b^2c^2}\right)W_1W_2\\
&+\left(\frac{2b\alpha}{a^2}+\frac{2\alpha\xi^2}{a^2bc^2}+\frac{2\alpha^3\xi}{ab^2c}\right)W_2W_4\\
&+\left(\frac{4\alpha\gamma}{a^2}+\frac{4\alpha\gamma\xi^2}{a^2b^2c^2}-\frac{2\beta b}{a^2}+\frac{2\beta\alpha^2\xi}{ab^2c^2}\right)W_1W_4\\
&+\left(\frac{4\gamma^3c}{a^2b^2}-\frac{2\gamma\beta\alpha\xi}{ab^2c}\right)W_1W_5+\left(-\frac{2\beta\alpha\xi}{abc^2}+\frac{2\beta\alpha^3}{b^2c^2}\right)W_1W_7\\
&+\left(\frac{2\gamma^2\beta}{ab^2}-\frac{2\beta^2\xi}{abc^2}+\frac{2\beta^2\alpha^2}{b^2c^2}\right)W_1W_6+\left(\frac{4\beta b}{a^2}+\frac{2\alpha^2\beta\xi}{ab^2c^2}\right)W_2W_3\\
&-\left(-\frac{4\beta\xi}{ac^2}+\frac{2\beta\alpha^2}{bc^2}+\frac{2\alpha\gamma\xi}{abc^2}\right)W_3W_7\\
&+\left(\frac{2\gamma\beta}{a^2}+\frac{2\gamma^3\beta}{a^2b^2}+\frac{2\beta^2\alpha\xi}{ab^2c^2}+\frac{4\gamma\beta\xi^2}{a^2b^2c^2}\right)W_1W_3\\
&+\left(\frac{4bc}{a^2}+\frac{4\xi^2}{a^2bc}+\frac{2\alpha^2\xi}{ab^2c}\right)W_2W_5-\frac{4cb_{,2}}{b}W_8\\
&+\left(\frac{2\beta\alpha^3}{b^2c^2}-\frac{2\gamma\alpha^2\xi}{ab^2c^2}-\frac{2\beta\alpha\xi}{a^2bc^2}\right)W_2W_6+\left(\frac{2\alpha\gamma}{ab}+\frac{2\alpha^4}{b^2c^2}\right)W_2W_7\\
&-\frac{6\alpha}{c}W_7W_8-\frac{2\alpha\xi}{abc}W_2W_8-\frac{2\beta\xi}{abc}W_3W_5\\
&-\left(\frac{2\gamma\beta\xi}{abc^2}+\frac{2\beta^2\alpha}{bc^2}\right)W_3W_6-\frac{2\beta\alpha\xi}{abc^2}W_3W_4+\left(\frac{4bc_{,3}}{c}-\frac{4\alpha b_{,2}}{b}\right)W_7\\
&+\left(\frac{2\alpha^3}{b^2c}-\frac{4\gamma\xi}{abc}\right)W_3W_8-\frac{4\alpha\xi}{abc}W_4W_5
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\frac{4\alpha\gamma\xi}{abc^2}-\frac{2\beta\alpha^2}{bc^2}-\frac{2\beta\xi}{abc}\right)W_4W_6+\left(\frac{2\alpha\xi}{ac^2}-\frac{2\alpha^3}{bc^2}\right)W_4W_7-\left(\frac{2\gamma\xi}{ac^2}+\frac{2\xi}{ac}\right)W_4W_8\\
&+\left(\frac{4\gamma\xi}{abc}-\frac{4\beta\alpha}{bc}\right)W_5W_6\\
&+\frac{4\xi}{ac}W_5W_7-\frac{2\alpha}{b}W_5W_8-\left(\frac{4\beta\alpha}{c^2}+\frac{2\beta}{c}\right)W_6W_7\\
&-\left(\frac{2\alpha\gamma}{bc}+\frac{2\beta}{c}+\frac{2\gamma}{b}\right)W_6W_8-4c\left(\frac{\gamma_{,2}}{a}-\frac{\gamma b_{,2}}{ab}\right)W_5\\
&+\left(\frac{2\beta(\gamma_{,3}+\beta_{,2})}{a}-\frac{2\beta_{,3}\gamma}{a}+\frac{4\gamma^2\alpha_{,3}}{ab}+\frac{4\gamma\xi c_{,3}}{abc}\right)W_1\\
&+\left(\frac{2\alpha(\gamma_{,3}+\beta_{,2})}{a}-\frac{4\beta_{,3}\gamma}{a}+\frac{4\gamma^2\alpha_{,3}}{ab}+\frac{4\gamma\xi c_{,3}}{ac}\right)W_2\\
&+\left(\frac{2b(\gamma_{,3}+\beta_{,2})}{a}-\frac{2\gamma(b_{,3}+\gamma_{,2})}{b}-4\alpha\left(\frac{\gamma_{,2}}{a}-\frac{\gamma b_{,2}}{ab}\right)\right)W_4\\
&+\left(\frac{4\gamma c_{,3}}{c}-\frac{4\beta b_{,2}}{b}\right)W_6\\
&+\left(\frac{2\gamma(\gamma_{,3}+\beta_{,2})}{a}-\frac{2\gamma^2(b_{,3}+\alpha_{,2})}{ab}-4\beta\left(\frac{\gamma_{,2}}{a}-\frac{\gamma\beta_{,2}}{ab}\right)\right)W_3=0,
\end{aligned}
\tag{C.2.2}
$$

where $\xi=\alpha\gamma-\beta b$.

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