



Constitutive modeling of Kelvin–Voigt type viscoelastic solids undergoing finite deformations using the theory of evolving natural configurations

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ARTICLE INFO

Keywords:

Viscoelasticity
Finite deformation
Kelvin–Voigt model
Thermodynamics
Polymers

ABSTRACT

The theory of evolving natural configurations is an effective technique to model dissipative processes. In this paper, we use this theory to develop nonlinear constitutive models of Kelvin–Voigt type viscoelastic solids within a Lagrangian framework. The constitutive models are developed using a stress-space formulation that incorporates the rate of dissipation as a convex function of the relevant configurational forces. Furthermore, we show that the basic Maxwell and Kelvin–Voigt models can be obtained as limiting cases from the derived Poynting–Thompson and Zener models, respectively. Integration algorithms for the developed models are presented and the numerical solution for a relevant boundary value problem is obtained to show the efficacy of the developed models. The response of the developed isotropic Poynting–Thompson model has been compared and benchmarked with the available experimental data. The presented model shows a very good match with the experimental data obtained from a uniaxial tensile test of polymers over a large extent of stretch (up to 9) and a wide range of stretch rate (0.2 s^{-1} to 400 s^{-1}). The relaxation behavior and rate effects for the Poynting–Thompson model has been studied. Finally, an extension of the Poynting–Thompson model to other material symmetry groups has been discussed.

1. Introduction

Rheological models are often used to describe the response of viscoelastic materials within a small-strain theory. These models typically consist of two types of basic rheological elements: a spring, representing the non-dissipative elastic response, and a viscous damper accounting for the rate-dependent, dissipative response of the material. Different arrangements of these elements either in a series or a parallel connection have long been used to model the response of different aspects of viscoelastic materials such as stress relaxation, creep, fading memory effects etc. These models, however, are developed within a small strain theory which precludes their use for a large class of materials. For example, soft elastomers and biological tissues often exhibit viscoelastic response, but their deformation often goes well beyond the small strain limit. Recent advances in the extensive use of these materials have propelled a need for the extension of these rheological models to a thermodynamically consistent finite deformation framework.

Recognizing this need, many attempts have been made to develop thermodynamically consistent nonlinear viscoelastic models. In particular, finite deformation versions of a Maxwell model (Haupt and Lion, 2002), its generalized form (Govindjee and Simo, 1992) and the associated standard solid model (Lubliner, 1985; Holzapfel, 1996; Zhou et al., 2018; Feng et al., 2021) garnered much attention. To develop

these models, a multiplicative decomposition of the deformation gradient (Bilby et al., 1955; Kröner, 1959; Lee, 1969) is typically used, in conjunction with an additive split of the Helmholtz free energy into its equilibrium and non-equilibrium parts. Thereafter, the evolution equation is derived from the condition that the resulting dissipation must be non-negative. These models have been formulated from the current (Reese and Govindjee, 1998), intermediate (Lion, 1997) as well as the reference configuration (Le Tallec et al., 1993) (see Gouhier and Diani, 2024 for a comparison between these formulations). Using the same modeling strategy, the problem was also recast into a framework involving two potential functions, viz., a Helmholtz free energy accounting for the energy stored in the material and a dissipation potential (Kumar and Lopez-Pamies, 2016; Sadik and Yavari, 2024). It is also noted here that other modeling approaches for viscoelastic materials also exist that are not based on the idea of rheological models. For example, Anssari-Benam and Hossain (2023) incorporated rate-effects into hyperelastic models by considering the material parameters to evolve with strain rate. Viscoelastic models based on a free energy and a dissipation potential are also available in the literature in which the choices of their functional forms are not derived from any rheological model (Limbert and Middleton, 2004; Anssari-Benam et al., 2018).

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While Maxwell-type models have been widely used and implemented into commercial finite element packages, their Kelvin–Voigt counterparts have received considerably less attention. This is due to the fact that unlike the Maxwell-type models, a Kelvin–Voigt model does not naturally fit into the framework of a multiplicative decomposition of the deformation gradient. Nevertheless, a Kelvin–Voigt type standard solid (Poynting–Thompson) model has been developed by Huber and Tsakmakis (2000) by extending the idea of additive decomposition of the Helmholtz free energy. With an appropriate choice for the Helmholtz free energy, they derived the evolution equation through an ad hoc ansatz that satisfies the requirement of a Clausius–Duhem inequality. An alternative framework for Kelvin–Voigt model has also been used in which an additive split of the Cauchy or Piola–Kirchhoff stress tensor is adopted instead of the multiplicative decomposition of the deformation gradient. This modeling technique bears resemblance to Simo’s (1987) overstress model which has been widely used in finite viscoelasticity (Kalske and Rothert, 1997; Holzapfel et al., 2002). Although this modeling technique has been extensively adopted and made its way into commercial finite element (FE) packages, its thermodynamic consistency is still debated (Govindjee et al., 2014; Liu et al., 2021). A possible explanation for the veracity of this model was provided by Rajagopal (2009) where he considered the Kelvin–Voigt model as a mixture between an elastic solid and a Newtonian fluid with equal volume fractions. The constitutive relation for the Kelvin–Voigt model, however, was obtained based on this rationale without any formal derivation. Based on this idea, different variants of the Kelvin–Voigt model have been proposed particularly in the context of implicit constitutive theories (Coco and Saccomandi, 2023; Itou et al., 2023; de Castro Motta et al., 2025).

Recently, Singh and Paul (2025) pointed out that the issues of thermodynamic inconsistency is not limited to a Kelvin–Voigt model, but persists in any rheological model associated with a parallel connection between the rheological elements within a finite deformation framework. A potential remedy for this problem was proposed using the theory of evolving natural configurations, developed by Rajagopal (1995) based on the idea of Eckart (1948). In this theory, it is assumed that a body can possess natural configurations that are obtained through a local elastic unloading from the current configuration of the body. Akin to Kumar and Lopez-Pamies (2016) and Sadik and Yavari (2024), this theory also requires specification of the Helmholtz free energy, ψ and a rate of dissipation function, ξ . While the former accounts for the elastic response of the body measured from a fixed natural configuration, the latter governs the evolution of the natural configurations. The primary advantage of this theory lies in the determination of the evolution equation. Unlike the traditional Coleman–Noll procedure, here only non-negative values of ξ are stipulated as admissible which ensures that a Clausius–Duhem form of the second law of thermodynamics is identically satisfied. Thereafter, the evolution equation is determined through a constrained maximization of the rate of dissipation function (Rajagopal and Srinivasa, 2004; Ziegler, 1963). This treatment allows one to choose the rate of dissipation function either in terms of the kinematic variables or the configurational forces responsible for the evolution of the natural configurations. We refer to these techniques as a strain-space and stress-space formulation, respectively. The strain-space formulation has previously been used in obtaining the finite deformation versions of several “rate-type” fluid models such as a Maxwell and an Oldroyd model (Rajagopal and Srinivasa, 2001), a Burgers model (Málek et al., 2015) etc. While the stress-space formulation is shown to be effective in the context of plasticity to retrieve a J_2 plasticity (Rajagopal and Srinivasa, 1998a, 2004) or a Drucker–Prager model (Paul and Freed, 2021; Paul, 2025), its use in finite viscoelasticity has been rather limited.

The primary objective of this paper is to develop thermodynamically consistent finite-deformation constitutive models for Kelvin–Voigt type viscoelastic materials that capture the underlying physical mechanisms of their rheological counterparts without resorting to a decomposition

of the Cauchy or Piola–Kirchhoff stress into its elastic and viscous parts. In a 1-D Kelvin–Voigt type rheological model, the *available* stress (or driving force) in the dissipative element (dashpot) is first determined from equilibrium and thereafter, the evolution equation for strain in the dashpot is expressed in terms of this available stress. Therefore, a Kelvin–Voigt type solid undergoes a dissipative process in such a way that the available stress (or, driving force) in the dashpot acts as an independent state variable (“determinant” or cause) whereas the strain in this element is a dependent function (“resultant” or effect). On the other hand, this causality is reversed in a Maxwell type rheological model. In the latter, the stress in the dashpot is determined as a function of the strain rate which, in turn, is obtained from a suitable compatibility condition. Hence, the strain in a dissipative element acts as an independent, primary state variable or determinant in this model. This is further corroborated by the fact that in a Maxwell model, the total mechanical power is distributed into the rate of elastic strain energy and the rate of dissipation in accordance with the ratio of strain rates in the spring and the dashpot whereas the energy distribution in a Kelvin–Voigt model follows the ratio of stresses (Singh and Paul, 2025). To develop a thermodynamically consistent material model, identifying the correct causal relationship is a crucial step to retain the underlying physics even though the resulting constitutive relation and its thermodynamic dual may be mathematically equivalent under certain conditions. This viewpoint has been rigorously argued by Rajagopal and Srinivasa (2019). Although a correct causal relationship has been preserved in the derivation of Maxwell-type finite viscoelastic models available in the literature (e.g., Kumar and Lopez-Pamies, 2016; Sadik and Yavari, 2024), the same is generally not true for traditional Helmholtz free energy-based finite Kelvin–Voigt type models. For the latter, either the physical interpretation of the dissipative structure has remained obscured or a reversed causality has been used in its derivation. For example, in the Poynting–Thompson model derived by Huber and Tsakmakis (2000), the viscous rate-of-deformation tensor is expressed in terms of kinetic variables (overstress tensors) which suggests that the underlying causal relationship has been maintained. However, despite the correct mathematical structure, the physical meanings of these kinetic variables as well as their role in the dissipation mechanism lack adequate clarity.¹ On the other hand, in a subsequent model proposed by Laiarinandrasana et al. (2003), the dissipation potential for a Poynting–Thompson solid was considered to be a function of the rate of viscous right Cauchy–Green tensor implying that the *kinematic* variable was considered to be an independent state variable or determinant. Even though the mathematical structures of both these models are consistent, the physical arguments used in their derivation remains unclear or questionable. The aforementioned finite Kelvin–Voigt models (Coco and Saccomandi, 2023; de Castro Motta et al., 2025), developed using an additive decomposition of the Cauchy stress tensor and a Gibbs potential-based approach, are notable exceptions that preserve the causality of its rheological counterpart. This paper is aimed at a possible resolution of this issue by developing constitutive models for finite Kelvin–Voigt type solids that preserve the causality of their rheological counterparts without resorting to an additive decomposition of the stress tensor. For this purpose, while Maxwell-type models are derived using a strain-space formulation following the traditional viewpoint of splitting the free energy into equilibrium and non-equilibrium parts, a stress-space formulation is used for the development of Kelvin–Voigt type material models. In the latter, the rate of dissipation is written as a convex function of the configurational forces ensuring that these forces are considered as a primary state variable or determinant that govern the dissipative process. Thus, our approach advocates the central role of

¹ The physical meaning of the governing equation for their Poynting–Thompson solid model may be understood in view of Eqs. (38) and (40) following the discussion in Section 3.3.

the configurational forces in determining the evolution equation and the overall response of a Kelvin–Voigt type material and consequently, the desired causality of its rheological counterpart is preserved. To this end, our paper is a departure from the traditional viewpoint and establishes a direct link between causality and the nature of the underlying rheological connections for a class of finite viscoelastic models.

The rest of the paper is organized as follows. In Section 2, we briefly review the theory of an evolving natural configurations. Particularly, a strain-space formulation of this theory and its applications to Maxwell and the associated standard solid model have been briefly revisited. Here we derive the models within a Lagrangian framework as opposed to the Eulerian formulation of the rate-type fluids of [Rajagopal and Srinivasa \(2001\)](#). In Section 3, we first derive the required configurational force (Eshelby energy–momentum tensor) for a chosen functional form for the Helmholtz free energy. Based on this configurational force, a set of general constitutive relations for compressible and incompressible materials is formulated. A constitutive model for isotropic Poynting–Thompson solids is then derived from this formulation in Section 3.3. In Section 3.3.2, the finite deformation version of a Kelvin–Voigt model is derived as a limiting case of the derived Zener model. The derived Kelvin–Voigt model is found to be consistent with the original idea of [Rajagopal \(2009\)](#). Although the primary aim of this paper is to extend Kelvin–Voigt type viscoelastic models to finite deformations, we also develop numerical solutions for relevant boundary value problems in Section 4 to demonstrate the efficacy of the proposed models and verify them with available experimental data. In Section 5, we discuss the extensions of the presented Poynting–Thompson models to other material symmetry groups. Finally, the paper is drawn to conclusion with a summary of the present work and possible future directions.

2. Preliminaries

We briefly revisit the theory of evolving natural configurations developed by [Rajagopal \(1995\)](#) and [Rajagopal and Srinivasa \(1998a\)](#), which serves as an essential ingredient in our proposed models. In this section, we only highlight the strain-space formulation and its application to derive Maxwell-type models.

2.1. Kinematics

Let us consider a simply connected homogeneous body $\mathcal{B}$ embedded in an Euclidean point space. At time $t = 0$, the embedding of the body is identified with a subset of this point space, called the undeformed configuration of the body, $\kappa_r(\mathcal{B})$. Let us denote the position vector of a material particle in $\kappa_r(\mathcal{B})$ by $\mathbf{X}$. A motion of the body is a smooth bijective map $\mathbf{x} = \mathcal{X}(\mathbf{X}, t)$ that maps it to the position vector of the same particle in its current configuration $\kappa_t(\mathcal{B})$ at time t , denoted by $\mathbf{x}$. Let $d\mathbf{X}$ and $d\mathbf{x}$ denote an infinitesimal fiber in the undeformed and current configuration of the body, respectively. The deformation gradient, $\mathbf{F}$, is a tangent map that takes the infinitesimal fiber $d\mathbf{X}$ from the tangent space of $\kappa_r(\mathcal{B})$ and places it into the tangent space of $\kappa_t(\mathcal{B})$, i.e., $d\mathbf{x} = \mathbf{F}d\mathbf{X}$. We further require that the Jacobian of the map $J = \det(\mathbf{F}) > 0$. In terms of the motion, the deformation gradient $\mathbf{F}$ can be written as

$$\mathbf{F} = \nabla_{\mathbf{X}} \mathcal{X}(\mathbf{X}, t). \quad (1)$$

The right and left Cauchy Green tensor, denoted by $\mathbf{C}$ and $\mathbf{B}$, respectively, are defined as

$$\mathbf{C} := \mathbf{F}^T \mathbf{F}, \quad \text{and} \quad \mathbf{B} := \mathbf{F} \mathbf{F}^T. \quad (2)$$

Based on the right Cauchy–Green tensor that serves as a metric for the undeformed configuration of the body, $\kappa_r(\mathcal{B})$, the Green strain tensor is defined as

$$\mathbf{E} := \frac{1}{2} (\mathbf{C} - \mathbf{I}). \quad (3)$$

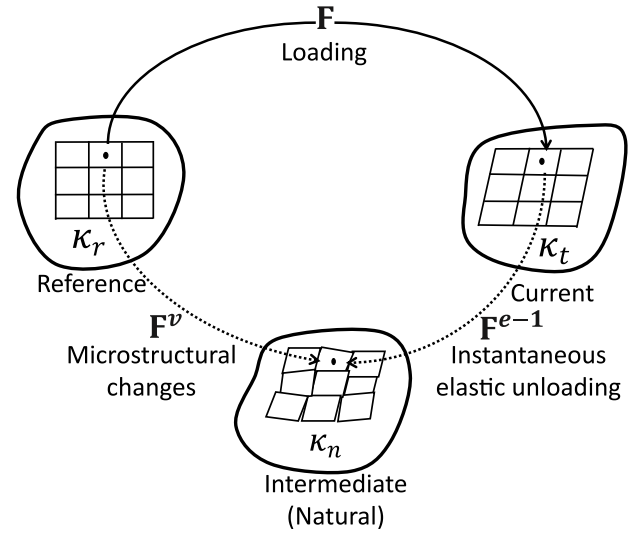


Fig. 1. Evolving natural configurations with the associated tangent maps.

To model the viscoelastic behavior of materials, we appeal to the theory of evolving natural configurations, developed in a series of papers by [Rajagopal and Srinivasa \(1998b, 2004\)](#). According to this theory, if an infinitesimal neighborhood of a particle in the current configuration of the body is instantaneously elastically unloaded through a tangent map $\mathbf{F}^{e-1}$ (i.e., all the external stimuli are instantaneously removed) at a particular time instant, the neighborhood around that particle occupies a local natural configuration, denoted by $\kappa_n(\mathcal{B})$. An infinitesimal fiber in the natural configuration of the body $\kappa_n(\mathcal{B})$ is related to the corresponding fiber in its reference configuration through another tangent map $\mathbf{F}^v$ such that

$$\mathbf{F}^v := \mathbf{F}^{e-1} \mathbf{F}. \quad (4)$$

In general, the natural configuration of the body is not globally compatible as shown in Fig. 1. Therefore, unlike the total deformation gradient $\mathbf{F}$, the tangent maps $\mathbf{F}^e$ and $\mathbf{F}^v$, in general, cannot be obtained as gradients of a deformation map. This is one of the reasons for which, despite Eq. (4) being mathematically equivalent to the extensively used multiplicative decomposition of the deformation gradient, i.e., $\mathbf{F} = \mathbf{F}^e \mathbf{F}^v$, the physical basis of the latter remains debatable (cf. [Green and Naghdi, 1971](#); [Casey and Naghdi, 1980](#); [Naghdi, 1990](#)). On the other hand, the theory of evolving natural configurations provides the necessary physical basis to resolve this issue. Here $\mathbf{F}^e$ and $\mathbf{F}^v$ are considered as linear maps between tangent vectors (infinitesimal fibers) in different configurations without the requirement of these maps being gradients of a motion. It should be also noted here that unlike anelasticity, the local natural configuration is not stress-free. In fact, the local natural configuration is subjected to residual stresses and the body occupies only a stress-free configuration upon further viscous relaxation. Therefore, a viscoelastic material can be viewed as a body that has evolving natural configurations.

Now from the associated tangent maps between the reference, natural and current configurations of the body, the corresponding right and left Cauchy–Green tensor-like kinematic variables can be defined as

$$\mathbf{C}^e := \mathbf{F}^{eT} \mathbf{F}^e, \quad \mathbf{C}^v := \mathbf{F}^{vT} \mathbf{F}^v, \quad \mathbf{B}^e := \mathbf{F}^e \mathbf{F}^{eT}, \quad \mathbf{B}^v := \mathbf{F}^v \mathbf{F}^{vT}. \quad (5)$$

For our analysis, it will be useful to define the kinematic variables in terms of strains similar to the Green strain tensor, $\mathbf{E}$. We define the strain tensors corresponding to the tangent maps $\mathbf{F}^e$ and $\mathbf{F}^v$ as

$$\mathbf{E}^e := \frac{1}{2} (\mathbf{C}^e - \mathbf{I}), \quad \text{and} \quad \mathbf{E}^v := \frac{1}{2} (\mathbf{C}^v - \mathbf{I}). \quad (6)$$

A routine calculation yields the relationship between the total strains and the Green strain-like kinematic variables associated with $\mathbf{F}^e$ and $\mathbf{F}^v$ as (Lubarda and Krajcinovic, 1995)

$$\mathbf{E} = \mathbf{E}^v + \mathbf{F}^{vT} \mathbf{E}^e \mathbf{F}^v. \quad (7)$$

2.2. Constitutive relations

In the theory of evolving natural configurations, constitutive relations are derived by specifying two thermodynamic quantities— a Helmholtz free energy, ψ and a rate of dissipation function, ξ . From the first law of thermodynamics, we obtain the relationship between these two quantities as

$$\xi = \mathbf{S} : \dot{\mathbf{E}} - \rho_0 \dot{\psi} \geq 0 \quad (8)$$

where $\mathbf{S}$ is the second Piola Kirchhoff stress. Now let us assume the functional forms for the Helmholtz free energy and the rate of dissipation. Since the Helmholtz free energy ψ accounts for the elastic response of the body measured from a fixed natural configuration, it is reasonable to assume ψ as a function of the elastic part of the deformation gradient $\mathbf{F}^e$, i.e., $\psi = \bar{\psi}(\mathbf{F}^e) = \bar{\psi}(\mathbf{F}, \mathbf{F}^v)$ for an isothermal process. Now in view of the material frame indifference, the pairs $(\mathbf{C}, \mathbf{C}^v)$ or alternatively, $(\mathbf{E}, \mathbf{E}^v)$ can be used in the argument of ψ . Here we choose the latter for convenience. In other words, we consider the Helmholtz free energy to be a function of the elastic Green strain-like tensor, pulled back into the undeformed configuration $\kappa_r(\mathcal{B})$ via $\bar{\mathbf{E}}^e = \mathbf{F}^{vT} \mathbf{E}^e \mathbf{F}^v$. Thus, in view of Eq. (7), the Helmholtz free energy can be written as

$$\psi = \bar{\psi}(\mathbf{E}, \mathbf{E}^v) = \bar{\psi}(\bar{\mathbf{E}}^e). \quad (9)$$

On the other hand, the rate of dissipation function accounts for the evolution of the natural configuration. Therefore, for a strain-space formulation, its functional form is assumed as

$$\xi = \bar{\xi}(\mathbf{E}^v, \dot{\mathbf{E}}^v) \quad \text{with} \quad \bar{\xi}(\mathbf{E}^v, \mathbf{0}) = 0. \quad (10)$$

The last part of Eq. (10) implies that whenever there is no change in the natural configuration, i.e., $\dot{\mathbf{E}}^v = \mathbf{0}$, the rate of dissipation should also be zero resulting in a purely elastic process. We consider that the elastic response of the body for a given natural configuration is that of a Green elastic solid such that

$$\mathbf{S} = \rho_0 \frac{\partial \bar{\psi}}{\partial \mathbf{E}}. \quad (11)$$

Using Eq. (11) and the functional forms of ψ and ξ in Eqs. (9) and (10) into the balance of energy Eq. (8), we obtain a thermodynamic constraint for the rate of dissipative function which reads

$$\xi = -\rho_0 \frac{\partial \bar{\psi}}{\partial \mathbf{E}^v} : \dot{\mathbf{E}}^v. \quad (12)$$

The evolution equation for $\mathbf{E}^v$ is now derived following the procedure given by Rajagopal and Srinivasa (2004). In a traditional Coleman-Noll procedure (Noll, 1967), an expression for the rate of dissipation is obtained from the balance of energy and thereafter, the evolution equation for the internal variable such as $\mathbf{E}^v$ is obtained in such a way that the non-negativity of the rate of dissipation is ensured. Rajagopal and Srinivasa (2004) adopted an alternative approach where only those kinematic variables that render the rate of dissipation function non-negative are considered to be admissible. The evolution equation is then obtained by employing a criterion of maximum entropy production. This procedure poses the derivation of the evolution equation as a constrained optimization problem where the rate of dissipation is maximized subject to the constraint (12). This departure has a major consequence in our derivation of a Kelvin–Voigt type material models which will be discussed later. For a strain-space formulation, the viscous strain rate is allowed to vary with prescribed driving forces and satisfying the constraint (12). Following this procedure, the

evolution equation of $\mathbf{E}^v$ for a compressible material can be derived via a Lagrange multiplier technique as

$$-\rho_0 \frac{\partial \bar{\psi}}{\partial \mathbf{E}^v} = \frac{1 + \lambda}{\lambda} \frac{\partial \bar{\xi}}{\partial \dot{\mathbf{E}}^v} \quad (13)$$

where λ is a Lagrange multiplier which can be obtained by the satisfaction of the constraint Eq. (12).

For an incompressible material, additional constraints associated with the change in mass density (or volume) must be incorporated. In a Lagrangian framework, this constraint is written as $\det(\mathbf{F}) = J = 1$. In addition to the total volume, it is reasonable to assume that there will be no change in the volume during the viscous relaxation process, i.e., $\det(\mathbf{F}^v) = J^v = 1$. These two constraints enter into the constitutive relations through a modification of the Helmholtz free energy as $\psi_{inc} = \psi - p(J - 1) - q(J^v - 1)$ where $p(\mathbf{X}, t)$ and $q(\mathbf{X}, t)$ are Lagrange multipliers (Sadik and Yavari, 2024). Therefore, from Eq. (11), the second Piola–Kirchhoff stress can be written as

$$\mathbf{S} = \rho_0 \frac{\partial \bar{\psi}}{\partial \mathbf{E}} - p J \mathbf{C}^{-1}. \quad (14)$$

Since the rate of dissipation is a function of $\mathbf{E}^v$ and $\dot{\mathbf{E}}^v$, it needs to be maximized with an additional constraint of $J^v - 1 = 0$. Therefore, the Lagrangian for maximizing the rate of dissipation reads

$$\mathcal{L}_{inc} = \bar{\xi} + \lambda \left[\bar{\xi} + \left(\rho_0 \frac{\partial \bar{\psi}}{\partial \mathbf{E}^v} - q J^v \mathbf{C}^{v-1} \right) : \dot{\mathbf{E}}^v \right]. \quad (15)$$

Now maximizing the Lagrangian in Eq. (15) with respect to $\dot{\mathbf{E}}^v$ and noting that J^v is a function of $\mathbf{E}^v$ only, the evolution Eq. (13) takes on a new form as

$$-\rho_0 \frac{\partial \bar{\psi}}{\partial \mathbf{E}^v} = \frac{1 + \lambda}{\lambda} \frac{\partial \bar{\xi}}{\partial \dot{\mathbf{E}}^v} - q J^v \mathbf{C}^{v-1}. \quad (16)$$

One can easily notice the similarity between our evolution equation and the kinetic equation for incompressible materials derived by Le Tallec et al. (1993) and Sadik and Yavari (2024).

2.3. Maxwell-type materials

As mentioned earlier, the theory of evolving natural configurations have been previously used to derive constitutive models for rate-type fluids that include Maxwell, Oldroyd-B and Burgers model (Rajagopal and Srinivasa, 2001; Málek et al., 2015). Before going into the development of Kelvin–Voigt type material models, we briefly revisit the Maxwell-type materials that will be used in our subsequent derivation. Here we present a Lagrangian formulation of the Maxwell model as well as its associated standard solid (Zener) model as opposed to the Eulerian formulation that has been widely developed earlier.

2.3.1. Maxwell model

To develop our constitutive model, we first need to select an appropriate functional form for the Helmholtz free energy and the rate of dissipation function. Here we confine our attention to only isotropic materials. Typically, the invariants of the right or left Cauchy–Green tensor are used in the functional form for the Helmholtz free energy, e.g., a Neo-Hookean or a Mooney–Rivlin material model. However, since there is a one-to-one correspondence between the invariants of the Green strain tensor and that of the Cauchy–Green tensors (Gurtin et al., 2010), it is reasonable to use the invariants of the Green strain tensor or its elastic and viscous parts to construct the Helmholtz free energy (Anssari-Benam et al., 2024). Here the Helmholtz free energy is chosen to be a function of the three invariants of the elastic Green strain, $\bar{\mathbf{E}}^e$. These invariants are given as

$$I_1 = \text{tr}(\bar{\mathbf{E}}^e), \quad I_2 = \text{tr}(\bar{\mathbf{E}}^{e^2}), \quad \text{and} \quad I_3 = \text{tr}(\bar{\mathbf{E}}^{e^3}). \quad (17)$$

In terms of these invariants, the Helmholtz free energy can be written as

$$\psi = \psi(I_1, I_2, I_3), \quad \text{with} \quad \alpha_i = \rho_0 \partial \psi / \partial I_i, \quad i = 1, 2, 3. \quad (18)$$

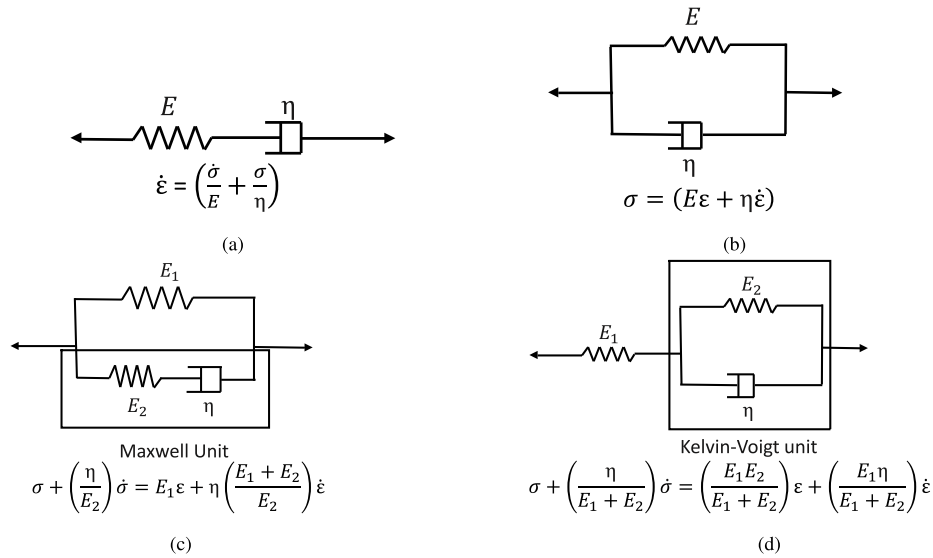


Fig. 2. The rheological networks of a linear: (a) Maxwell (b) Kelvin-Voigt (c) Zener and, (d) Poynting-Thompson model.

The rate of dissipation function is chosen as a quadratic function of $\dot{\mathbf{E}}^v$ in accordance with the postulates of Singh and Paul (2025). Thus, the rate of dissipation function reads

$$\xi = \eta \dot{\mathbf{E}}^v : \dot{\mathbf{E}}^v \quad (19)$$

where $\eta > 0$ is the viscosity of the material. Clearly, the rate of dissipation is non-negative and also a convex function in $\dot{\mathbf{E}}^v$. Now substituting the chosen Helmholtz free energy in Eq. (18) into Eq. (11) and using Eq. (7), we obtain the second Piola-Kirchhoff stress as

$$\mathbf{S} = \tilde{\mathbf{S}} + 3\alpha_3 (\mathbf{E} - \mathbf{E}^v)^2 \quad \text{with} \quad \tilde{\mathbf{S}} = \alpha_1 \mathbf{I} + 2\alpha_2 (\mathbf{E} - \mathbf{E}^v). \quad (20)$$

Here $\tilde{\mathbf{S}}$ acts as the deviatoric part of the second Piola-Kirchhoff stress. The evolution equation for $\mathbf{E}^v$ is obtained by substituting the rate of dissipation function into (13) as

$$\dot{\mathbf{E}}^v = \frac{1}{\eta} \left(\alpha_1 \mathbf{I} + 2\alpha_2 (\mathbf{E} - \mathbf{E}^v) + 3\alpha_3 (\mathbf{E} - \mathbf{E}^v)^2 \right). \quad (21)$$

One can easily notice the similarity between the (linearized) evolution equations of Lubliner (1985), Reese and Govindjee (1998) and Eq. (21) with some additional terms associated with the material parameters α_1 and α_3 . Now combining Eqs. (7), (20) and (21), we arrive at the governing equation for a Maxwell material as

$$\dot{\tilde{\mathbf{S}}} + (2\alpha_2/\eta) \mathbf{S} = 2\alpha_2 \dot{\mathbf{E}}. \quad (22)$$

It is worth noting that the model proposed by Bonet (2001) and Liu et al. (2025) emerges as a special case of Eq. (22) under the condition of a constant deformation gradient, i.e., when $\dot{\mathbf{E}} = 0$. We further note that the developed governing equation is slightly different from the one developed by Rajagopal and Srinivasa (2001) or the traditionally used upper-convected Maxwell model. This is due to the fact that while an objective stress rate appears in these governing equations, our developed model involves a simple material time-derivative of the second Piola-Kirchhoff stress. This is a direct consequence of using a Lagrangian framework. Since the kinetic and kinematic variables used here, such as the second Piola-Kirchhoff stress and the Green strain remain the same under a transformation of the material frame, their rates are also the same under the same transformation. On the other hand, the Cauchy stress transforms according to $\mathbf{T}^* = \mathbf{Q} \mathbf{T} \mathbf{Q}^T$, $\mathbf{Q} \in \mathcal{O}^+(3)$ where $\mathbf{T}^*$ and $\mathbf{T}$ represent the Cauchy stresses, measured by two observers whose frames of reference differ by a rigid body motion. Therefore, to ensure form invariance of the constitutive relationship under material frame transformation, an objective stress rate is required

for an Eulerian frame of reference (Jog, 2015). In contrast, a material time derivative of the second Piola-Kirchhoff stress or the Green strain is sufficient to preserve the constitutive form invariance for a Lagrangian formulation.

For an incompressible material, while the term associated with the third invariant of $\tilde{\mathbf{E}}^v$ needs to be dropped from the expression of ψ , two additional constraints of $J = 1$ and $J^v = 1$ must be incorporated. Therefore, the Helmholtz free energy and the rate of dissipation function take the form

$$\psi = \psi(I_1, I_2), \quad \text{and} \quad \xi = \eta \dot{\mathbf{E}}^v : \dot{\mathbf{E}}^v. \quad (23)$$

Substituting ψ and ξ from Eq. (23) into (14) and (16), we arrive at

$$\eta \dot{\tilde{\mathbf{S}}} + 2\alpha_2 \tilde{\mathbf{S}} = 2\alpha_2 \eta \dot{\mathbf{E}} + 2\alpha_2 q J^v \mathbf{C}^{v-1} \quad (24)$$

with

$$\tilde{\mathbf{S}} = \alpha_1 \mathbf{I} + 2\alpha_2 (\mathbf{E} - \mathbf{E}^v) \quad \text{and} \quad \mathbf{S} = \tilde{\mathbf{S}} - p J \mathbf{C}^{-1}. \quad (25)$$

The governing equation for an incompressible Maxwell model is similar to that of a compressible material with a notable exception of the last term in Eq. (24).

2.3.2. Zener model

We now extend this theory to derive the constitutive equations for a Zener model, shown in Fig. 2(c). For a compressible Zener model, the Helmholtz free energy is assumed to depend on three additional invariants of the total strain $\mathbf{E}$. This choice of the Helmholtz free energy is based on the idea of decomposing it into its equilibrium (function of $\mathbf{E}$) and non-equilibrium (function of $\tilde{\mathbf{E}}^v$) parts (Reese and Govindjee, 1998; Hossain et al., 2012; Kumar and Lopez-Pamies, 2016; Sadik and Yavari, 2024). The three additional invariants are defined as

$$I_1^s = \text{tr}(\mathbf{E}), \quad I_2^s = \text{tr}(\mathbf{E}^2), \quad \text{and} \quad I_3^s = \text{tr}(\mathbf{E}^3). \quad (26)$$

With these invariants, the Helmholtz free energy is written as

$$\psi = \psi(I_1, I_2, I_3, I_1^s, I_2^s, I_3^s) \quad (27)$$

with $\alpha_i = \rho_0 \partial \psi / \partial I_i$ and $\alpha_{i+3} = \rho_0 \partial \psi / \partial I_i^s$, $i = 1, 2, 3$. The rate of dissipation function is chosen to have the same form as that of a Maxwell model. Substituting Eqs. (19) and (27) into Eqs. (11) and (13), we obtain

$$\eta \dot{\tilde{\mathbf{S}}} + 2\alpha_2 \mathbf{S} = 2\alpha_2 \alpha_4 \mathbf{I} + 2(\alpha_2 + \alpha_5) \eta \dot{\mathbf{E}} + 4\alpha_2 \alpha_5 \mathbf{E} + 6\alpha_2 \alpha_6 \mathbf{E}^2 \quad (28)$$

where

$$\tilde{\mathbf{S}} = \mathbf{S} - 3\alpha_3 (\mathbf{E} - \mathbf{E}^v)^2 - 3\alpha_6 \mathbf{E}^2 \quad \text{with} \quad \tilde{\mathbf{S}} = (\alpha_1 + \alpha_4) \mathbf{I} + 2(\alpha_2 + \alpha_5) \mathbf{E} - 2\alpha_2 \mathbf{E}^v. \quad (29)$$

For an incompressible Zener solid, we follow the same strategy as before. The third invariants of $\mathbf{E}$ and $\mathbf{E}^v$ are dropped from the expression of the Helmholtz free energy which leads to

$$\psi = \psi(I_1, I_2, I_1^s, I_2^s). \quad (30)$$

Now using Eqs. (14) and (16), we obtain

$$\eta \dot{\tilde{\mathbf{S}}} + 2\alpha_2 \tilde{\mathbf{S}} = 2\alpha_2 \alpha_4 \mathbf{I} + 2\eta (\alpha_2 + \alpha_5) \dot{\mathbf{E}} + 4\alpha_2 \alpha_5 \mathbf{E} + 2\alpha_2 q J^v \mathbf{C}^{v^{-1}}. \quad (31)$$

Here the expressions for the second Piola–Kirchhoff stress and its deviatoric part are given as

$$\mathbf{S} = \tilde{\mathbf{S}} - p J \mathbf{C}^{-1} \quad \text{with} \quad \tilde{\mathbf{S}} = (\alpha_1 + \alpha_4) \mathbf{I} + 2(\alpha_2 + \alpha_5) \mathbf{E} - 2\alpha_2 \mathbf{E}^v. \quad (32)$$

3. A stress space formulation of the evolving natural configurations framework

In this section, we develop the stress-space formulation of the theory of evolving natural configurations following the works of [Rajagopal and Srinivasa \(1998a, 2000\)](#). As discussed earlier, this formulation is essential for our subsequent development of the nonlinear Kelvin–Voigt type models.

3.1. Configurational forces

For a stress space formulation, we start with calculating the force on an inhomogeneity (known as a configurational or a driving force) following Eshelby's procedure ([Eshelby, 1957](#)) which was recast into the evolving natural configurations framework by [Rajagopal and Srinivasa \(2005\)](#). In this method, the configurational force is determined by observing the change in the total strain energy in a body due to a small perturbation in the natural configuration. Since the Helmholtz free energy ψ represents the stored energy per unit volume of the undeformed configuration of the body, the total strain energy is given by

$$\Psi = \int_{\Omega} \rho_0 \bar{\psi}(\mathbf{E}, \mathbf{E}^v) dV. \quad (33)$$

Now let us provide a small perturbation in the natural configuration (i.e., $\mathbf{E}^v$) with an arbitrary tensor-valued function $\alpha \mathbf{H}$, where $\alpha \in \mathbb{R}$ and $\alpha \ll 1$. Thus, the strain energy per unit volume after perturbation is

$$\psi = \bar{\psi}(\mathbf{E}, \mathbf{E}^v + \alpha \mathbf{H}). \quad (34)$$

The change in the total strain energy due to the perturbation, when the terms of the order $\mathcal{O}(\alpha^2)$ and higher are neglected, reads

$$-\frac{\partial \Psi}{\partial \alpha} \Big|_{\alpha=0} = \int_{\Omega} \left(-\rho_0 \frac{\partial \bar{\psi}}{\partial \mathbf{E}^v} : \mathbf{H} \right) dV. \quad (35)$$

Since $\alpha \mathbf{H}$ signifies the perturbation in the natural configuration in Eq. (35), following Eshelby's (1957) definition, the driving (configurational) force behind the changes in the natural configuration is given by

$$\mathbf{A}^v := -\rho_0 \frac{\partial \bar{\psi}}{\partial \mathbf{E}^v}. \quad (36)$$

This configurational force can be shown to be equivalent to the Eshelby's energy–momentum tensor ([Rajagopal and Srinivasa, 2005; Paul et al., 2022](#)).

It is important to note here that the definition of the configurational force depends on the chosen functional form for the Helmholtz free energy, ψ . For example, [Rajagopal and Srinivasa \(1998a\)](#) chose the Helmholtz free energy of the form $\psi = \bar{\psi}(\mathbf{F}, \mathbf{F}^v)$ which led to the

expression for the configurational force as $\mathbf{A}^v = -\rho_0 (\partial \bar{\psi} / \partial \mathbf{F}^v) \mathbf{F}^{v\top}$. In their derivation of a Kelvin–Voigt type standard solid model, [Huber and Tsakmakis \(2000\)](#) used the form of ψ to be $\psi = \hat{\psi}(\mathbf{F}^e, \mathbf{F}^v)$. In this case, the expression for the configurational force can be determined through the same exercise with some important modifications. Let us choose a small perturbation in $\mathbf{F}^e$ and $\mathbf{F}^v$ both in such a way that the total deformation gradient remains unaltered. With these perturbations, the Helmholtz free energy reads

$$\psi = \hat{\psi}(\mathbf{F}^e - \alpha \mathbf{F}^e \mathbf{H}, \mathbf{F}^v + \alpha \mathbf{H} \mathbf{F}^v) \quad \text{with} \quad \alpha \ll 1. \quad (37)$$

One can easily verify that a multiplication of the perturbed elastic and viscous deformation gradients results in the total deformation gradient $\mathbf{F}$ if the term of the order $\mathcal{O}(\alpha^2)$ is neglected. Now the configurational force is obtained from the variation of the total strain energy as

$$-\frac{\partial \hat{\Psi}}{\partial \alpha} \Big|_{\alpha=0} = \int_{\Omega} \rho_0 \underbrace{\left(\mathbf{F}^{e\top} \frac{\partial \hat{\psi}}{\partial \mathbf{F}^e} - \frac{\partial \hat{\psi}}{\partial \mathbf{F}^v} \mathbf{F}^{v\top} \right)}_{\mathbf{A}^v} : \mathbf{H} dV. \quad (38)$$

3.2. Constitutive relations

Using the configurational forces derived above, we now formulate the required constitutive relations within the stress space formulation. For a compressible material, the constraint Eq. (12) for the rate of dissipation can be alternatively written as

$$\xi = \mathbf{A}^v : \dot{\mathbf{E}}^v. \quad (39)$$

In a strain-space formulation, the evolution equation was obtained through a constrained maximization of the rate of dissipation function with respect to the admissible strain rates $\dot{\mathbf{E}}^v$ keeping the configurational forces fixed. For a stress-space formulation, this process is altered in a significant way. In this case, the configurational force is considered to be the primary variable such that the rate of dissipation function can be written as $\xi = \bar{\xi}(\mathbf{E}^v, \mathbf{A}^v)$. The evolution equation is now obtained by maximizing the rate of dissipation with respect to the admissible configurational force while the strain rate $\dot{\mathbf{E}}^v$ is held fixed. Now following the exercise of a constrained maximization of the rate of dissipation function, similar to the one shown in Section 2.2, we obtain

$$\dot{\mathbf{E}}^v = \frac{1}{\bar{\lambda}} \frac{\partial \bar{\xi}}{\partial \mathbf{A}^v} \quad (40)$$

where $\bar{\lambda}$ is the Lagrange multiplier that can be determined by the satisfaction of the constraint Eq. (39) as

$$\bar{\lambda} = \frac{\mathbf{A}^v : \partial \bar{\xi} / \partial \mathbf{A}^v}{\bar{\xi}}. \quad (41)$$

Note that unlike the strain-space formulation, here we obtain an explicit evolution equation for the internal variable $\mathbf{E}^v$ in terms of the configurational force $\mathbf{A}^v$. This fact leads to a significant distinction between the two formulations in terms of the physical interpretation of the dissipative process.

Under certain constitutive assumptions, the evolution Eqs. (13) and (40), derived from a strain- and a stress-space formulation respectively, may turn out to be mathematically equivalent. Therefore, one may argue that Eq. (40) is essentially an alternative, dual formulation of the same evolution Eq. (13), obtained in a strain-space formulation. However, despite their possible mathematical equivalency under certain constitutive assumption, their physical interpretation still remains distinct. This viewpoint has been rigorously argued by [Rajagopal and Srinivasa \(2019\)](#). Adopting their terminology, in a strain-space formulation, the kinematic variable such as $\mathbf{E}^v$ is considered to be an independent state variable or “determinant”. Therefore, the evolution Eq. (13) has a direct physical meaning in which the driving force $\mathbf{A}^v$ is expressed in terms of the strain rate $\dot{\mathbf{E}}^v$. While it may be possible to find a dual for the rate of dissipation, $\bar{\xi}(\mathbf{A}^v)$ under certain conditions such that Eq. (13) is inverted to express the strain rate $\dot{\mathbf{E}}^v$ in terms of the

driving force, the inverted equation remains a derived result and hence, does not provide any distinct physical interpretation, as compared to Eq. (13) (Rajagopal and Srinivasa, 2019, Section 3.3). On the other hand, in a stress-space formulation, the configurational or driving force acts as a “determinant” or cause. Hence, Eq. (40) is the primary result in which the dissipation is governed by the configurational force, $\mathbf{A}^v$ and hence, the evolution of $\mathbf{E}^v$ is expressed in terms of it. Although this relation can be mathematically inverted under certain conditions (e.g., a quadratic functional form for ξ) such that the configurational force is expressed in terms of $\dot{\mathbf{E}}^v$ and hence, a similarity or even equivalence with Eq. (13) may be observed, these equations are not equivalent in their physical interpretations.

For an incompressible material, the constraints $J = 1$ and $J^v = 1$ must be incorporated through the modification of the Helmholtz free energy as $\psi_{inc} = \psi - p(J - 1) - q(J^v - 1)$. This leads to the modified definition of the configurational force as

$$\mathbf{A}_{inc}^v = -\rho_0 \frac{\partial \psi}{\partial \mathbf{E}^v} + q J^v \mathbf{C}^{v-1}. \quad (42)$$

Now following the same procedure as before, it can be shown that the evolution equation for incompressible material remains the same while the difference due to the constraints of incompatibility appears only in the definition of the configurational force.

3.3. Modeling of Kelvin–Voigt type materials

We now use the stress-space formulation to develop constitutive models for Kelvin–Voigt type solids. As discussed earlier, a significant difference between the formulation of Maxwell and Kelvin–Voigt type models arises from the choice of the independent state variable in modeling their respective dissipation mechanisms. First we develop constitutive equations for a Poynting–Thompson model. Thereafter, a Kelvin–Voigt model is obtained as a limiting case of the derived standard solid models.

3.3.1. Poynting–Thompson model

To develop a finite Poynting–Thompson model, the functional forms for the Helmholtz free energy and the rate of dissipation need to be chosen judiciously such that the underlying physical mechanism of its rheological counterpart is adequately captured. Observing the arrangement of the rheological elements in Fig. 2(d), it is reasonable to assume that the equilibrium part of the Helmholtz free energy is a function of $\bar{\mathbf{E}}^e$ whereas its non-equilibrium part is considered to be a function of the viscous strain $\mathbf{E}^v$ (Huber and Tsakmakis, 2000). Thus, the functional form for the Helmholtz free energy can be written as

$$\psi = \psi(I_1, I_2, I_3, I_1^k, I_2^k, I_3^k) \quad (43)$$

where I_1, I_2, I_3 are the invariants defined in Eq. (17). The additional invariants are given as

$$I_1^k = \text{tr}(\mathbf{E}^v), \quad I_2^k = \text{tr}(\mathbf{E}^{v^2}) \quad \text{and}, \quad I_3^k = \text{tr}(\mathbf{E}^{v^3}). \quad (44)$$

Here $\alpha_i = \rho_0 \partial \psi / \partial I_i$ and $\alpha_{i+3} = \rho_0 \partial \psi / \partial I_i^k, i = 1, 2, 3$. From the chosen form for the Helmholtz free energy, we can obtain the second Piola–Kirchhoff stress and the configurational force from Eqs. (11) and (36), respectively as

$$\mathbf{S} = \rho_0 \frac{\partial \psi}{\partial \mathbf{E}} = \alpha_1 \mathbf{I} + 2\alpha_2 (\mathbf{E} - \mathbf{E}^v) + 3\alpha_3 (\mathbf{E} - \mathbf{E}^v)^2 \implies \tilde{\mathbf{S}} = \alpha_1 \mathbf{I} + 2\alpha_2 (\mathbf{E} - \mathbf{E}^v) \quad (45)$$

and,

$$\mathbf{A}^v = -\rho_0 \frac{\partial \psi}{\partial \mathbf{E}^v} = \mathbf{S} - \alpha_4 \mathbf{I} - 2\alpha_5 \mathbf{E}^v - 3\alpha_6 \mathbf{E}^{v^2} \quad (46)$$

where the deviatoric stress $\tilde{\mathbf{S}}$ is defined as $\tilde{\mathbf{S}} := \mathbf{S} - 3\alpha_3 (\mathbf{E} - \mathbf{E}^v)^2$.

As evident from Eq. (21), the evolution equations for a Maxwell and a Zener model are implicit equations in $\dot{\mathbf{E}}^v$ where the driving

force is expressed in terms of the rate of the viscous strain. In case of a Poynting–Thompson solid, this causality relation is reversed. To analyze the rheological Poynting–Thompson solid shown in Fig. 2(d), the evolution equation for the strain across the Kelvin–Voigt unit is evaluated as a function of the “available” stress across the viscous damper (dashpot) and hence, the dissipative process in this model is governed by the “available” stress across the dashpot (c.f., Singh and Paul (2025) for a detailed discussion). To capture this mechanism correctly, we first identify the strain across the Kelvin–Voigt unit with the viscous strain $\mathbf{E}^v$ and the “available” stress across the damper with the configurational force $\mathbf{A}^v$ for a finite-deformation Poynting–Thompson model. To be consistent with the physical mechanism of its rheological counterpart, the evolution equation for $\mathbf{E}^v$ needs to be expressed in terms of the configurational force. For this purpose, the stress space formulation, discussed in Section 3.2, is particularly useful. Let us choose the rate of dissipation function in terms of the configurational force as

$$\xi = \frac{1}{\eta} \mathbf{A}^v : \mathbf{A}^v. \quad (47)$$

The chosen rate of dissipation is a quadratic function of the configurational force $\mathbf{A}^v$ with the viscosity parameter $\eta > 0$ and hence, it can easily be verified that the rate of dissipation function ξ is convex in the configurational force space. Moreover, it satisfies the requirements stated in Theorem 3 of Rajagopal and Srinivasa (1998b). The choice of the functional form of the rate of dissipation function is a significant departure from the existing viscoelastic models.

Now using Eqs. (40) and (47) on the chosen rate of dissipation function, one can obtain the evolution equation for the viscous strain. From Eq. (41), the Lagrange multiplier is evaluated as $\bar{\lambda} = 2$. Therefore, substituting the value of $\bar{\lambda}$ and the expression for the driving force $\mathbf{A}^v$ from Eq. (46) into Eq. (40), the evolution equation for $\mathbf{E}^v$ is obtained as²

$$\dot{\mathbf{E}}^v = \frac{1}{\eta} \left(\mathbf{S} - \alpha_4 \mathbf{I} - 2\alpha_5 \mathbf{E}^v - 3\alpha_6 \mathbf{E}^{v^2} \right). \quad (48)$$

Now taking a material time derivative of Eq. (45) and using Eqs. (7) and (48), we obtain the governing equation for a Poynting–Thompson solid as

$$\eta \dot{\tilde{\mathbf{S}}} + 2\alpha_2 \mathbf{S} = 2\alpha_2 \eta \dot{\mathbf{E}} + 4\alpha_2 \alpha_5 \mathbf{E}^v + 2\alpha_2 \alpha_4 \mathbf{I} + 6\alpha_2 \alpha_6 \mathbf{E}^{v^2}. \quad (49)$$

For an incompressible material, the terms associated with I_3 and I_3^k are dropped and thus, the expression for the Helmholtz free energy reads

$$\psi = \psi(I_1, I_2, I_2^k, I_2^k). \quad (50)$$

The second Piola–Kirchhoff stress and the configurational force can be obtained from this Helmholtz free energy using Eqs. (14) and (42) respectively, as

$$\mathbf{S} = \tilde{\mathbf{S}} - p J \mathbf{C}^{-1} \quad \text{with} \quad \tilde{\mathbf{S}} = \alpha_1 \mathbf{I} + 2\alpha_2 (\mathbf{E} - \mathbf{E}^v) \quad (51)$$

and,

$$\mathbf{A}^v = \tilde{\mathbf{S}} - \alpha_4 \mathbf{I} - 2\alpha_5 \mathbf{E}^v + q J^v \mathbf{C}^{v-1}. \quad (52)$$

² Although Eqs. (21) and (48) may be mathematically recast into the form $\mathbf{A}^v = \eta \dot{\mathbf{E}}^v$, one should not confuse these two equations to be the same. They are derived from two substantially different evolution equations (Eqs. (13) and (40) respectively) and hence, carry different underlying physical meanings. Moreover, the recasting of the evolution equations only holds true for the simplest choice of ξ for the isotropic materials made here. For the anisotropic material models derived in Section 5, the evolution equations for the Maxwell/Zener and the Poynting–Thompson model, even mathematically, cannot be recast into the same form.

Now choosing the same expression for the rate of dissipation function as in Eq. (47) along with the modified definition of the configurational force in Eq. (52) and using Eqs. (51), (7) and (40), we obtain

$$\eta \dot{\mathbf{S}} + 2\alpha_2 \dot{\mathbf{S}} = 2\alpha_2 \eta \dot{\mathbf{E}} + 4\alpha_2 \alpha_5 \mathbf{E}^v + 2\alpha_2 \alpha_4 \mathbf{I} - 2\alpha_2 q J^v \mathbf{C}^{v-1}. \quad (53)$$

Eq. (53) is the required governing equation for an incompressible Poynting–Thompson model.

Linearization and comparison with the Zener model

We now show that the developed Poynting–Thompson solid model (49) reduces to the well-known linearized version under the condition of a small displacement gradient, i.e., $\max_{\mathbf{x}} \|\partial \mathbf{u} / \partial \mathbf{x}\| = \mathcal{O}(\delta)$, $\delta \ll 1$. Under this assumption, the Green strain reduces to the linearized strain, $\epsilon = 1/2(\nabla \mathbf{u} + (\nabla \mathbf{u})^T)$ whereas the decomposition in Eq. (7) becomes $\epsilon = \epsilon^e + \epsilon^v$. Since the magnitude of the linearized strain and its elastic and viscous parts are small, any nonlinear terms associated with them can be neglected. Therefore, the second Piola–Kirchhoff stress and the configurational force in Eqs. (45) and (46) reduce to

$$\mathbf{s} = \bar{\mathbf{s}} = \alpha_1 \mathbf{I} + 2\alpha_2 \epsilon^e \quad \text{and} \quad \mathbf{a}^v = \mathbf{s} - \alpha_4 \mathbf{I} - 2\alpha_5 \epsilon^v. \quad (54)$$

Now following the same exercise, the evolution equation for the viscous strain is evaluated as

$$\dot{\epsilon}^v = \frac{1}{\eta} (\mathbf{s} - \alpha_4 \mathbf{I} - 2\alpha_5 \epsilon^v). \quad (55)$$

Now combining Eqs. (54) and (55) and using the relation $\epsilon = \epsilon^e + \epsilon^v$, we obtain

$$\eta \dot{\mathbf{s}} + 2(\alpha_2 + \alpha_5) \mathbf{s} = 2\alpha_2 \eta \dot{\epsilon} + 4\alpha_2 \alpha_5 \epsilon + (\alpha_2 \alpha_4 + \alpha_1 \alpha_5) \mathbf{I}. \quad (56)$$

In a similar manner, the Zener model (28) can be linearized to

$$\eta \dot{\mathbf{s}} + 2\alpha_2 \mathbf{s} = 2(\alpha_2 + \alpha_5) \eta \dot{\epsilon} + 4\alpha_2 \alpha_5 \epsilon + 2\alpha_2 \alpha_4 \mathbf{I}. \quad (57)$$

As expected, it can be observed from Eqs. (56) and (57) that the governing equations for both these models are similar except for the differences in the coefficients of the associated stress and strain rate terms. On the other hand, the governing equations for their finite deformation counterparts, i.e., Eqs. (28) and (49), show significant differences owing to the nonlinear strain terms involved in these models.

3.3.2. Limiting case: Kelvin-voigt model

From the Poynting–Thompson model, one can easily observe that for a limiting case of $\alpha_4, \alpha_5, \alpha_6 \rightarrow 0$, the governing Eq. (49) reduces to that of a Maxwell model, i.e., Eq. (22). This result stems from the fact that the limits under consideration render the assumed form for ψ as well as the resulting configurational force of a Poynting–Thompson model same as that of a Maxwell model. Motivated by this observation, we now examine the limiting case for a Zener model.

Let us consider the material parameters associated with the scalar functions α_1, α_2 and α_3 for the Zener model to be much higher than that of the other functions. For a small-strain rheological model (Fig. 2(c)), this limit is tantamount to the spring with modulus E_2 being too stiff such that it acts like a rigid element. Therefore, under this condition, the Zener model (Fig. 2(c)) shows an equivalent response to that of a Kelvin–Voigt model (Fig. 2(b)). Now let us explore the implication of this limit for the finite deformation Zener model derived in Section 2.3.2. Dividing both sides of the governing Eq. (28) of the Zener model by $2\alpha_2$ and considering the numerical value of α_2 to be very large, the governing equation reduces to

$$\mathbf{S} = \alpha_4 \mathbf{I} + 2\alpha_5 \mathbf{E} + 3\alpha_6 \mathbf{E}^2 + \eta \dot{\mathbf{E}}. \quad (58)$$

Eq. (58) is the desired finite deformation version of the Kelvin–Voigt model in its Lagrangian form. It can easily be observed that Eq. (58) reduces to the well-known linear Kelvin–Voigt model under the assumption of a small displacement gradient. In his model, Rajagopal (2009) considered the Kelvin–Voigt material to be a mixture between

a neo-Hookean elastic solid and a Newtonian fluid which led to the constitutive assumption of

$$\mathbf{T} = \mu \mathbf{B} - p \mathbf{I} + \eta \mathbf{D}. \quad (59)$$

One can easily notice the similarity between (59) and our model with one important distinction. From the governing Eq. (58), our Kelvin–Voigt model can be interpreted as a mixture between a more general Green elastic solid (since $\alpha_{i+3} = \rho_0 \partial \psi / \partial I_i^s, i = 1, 2, 3$) and a Newtonian fluid. It is worth noting here that our Kelvin–Voigt model is obtained through appropriate choices of the Helmholtz free energy and the rate of dissipation function within the theory of evolving natural configurations under certain limit and its interpretation as a mixture is not necessary for its derivation. We conclude this section by providing the governing equation for an incompressible Kelvin–Voigt material. In view of the above interpretation, $J = 1$ can be considered as the only constraint of incompressibility here. Thus, from Eq. (31), we obtain the required governing equation as

$$\mathbf{S} = \alpha_4 \mathbf{I} + 2\alpha_5 \mathbf{E} - p J \mathbf{C}^{-1} + \eta \dot{\mathbf{E}}. \quad (60)$$

While the constitutive equations for the other viscoelastic models are implicit in nature, the constitutive equation for the Kelvin–Voigt model can be directly integrated as it is an explicit equation in the total deformation gradient, $\mathbf{F}$, or the resulting Green strain and its rate.

4. Integration algorithms and numerical examples

To show the efficacy of our derived constitutive models, we develop integration algorithms and demonstrate the utility of these models through relevant boundary value problems in this section. Although the development of a full-scale finite element framework is required to exploit the full potential of the developed models, such framework is beyond the scope of the current work and will be addressed in the future.

4.1. Numerical implementation of the developed models

As mentioned earlier, the governing equations for the Maxwell, Zener and Poynting–Thompson models are implicit in nature whereas for a Kelvin–Voigt model, it is an explicit equation in the Green strain $\mathbf{E}$ and its material time-derivative. Therefore, while the governing equation for a Kelvin–Voigt model can be directly integrated, a suitable numerical method must be adopted for the other three. In our analysis so far, we have considered a functional form for the Helmholtz free energy, ψ . To develop numerical solutions, however, we require a more specific form for this function. For this purpose, let us consider a Taylor series expansion of the Helmholtz free energy up to the second-order terms, such that the Helmholtz free energies, ψ corresponding to the Maxwell, Zener and Poynting–Thompson models, take the form (Du et al., 2024)

$$\rho_0 \psi_{MM} = \psi_{eq}(I_1, I_2, I_3) = \frac{\lambda}{2} I_1^2 + \mu I_2 \quad (61a)$$

$$\rho_0 \psi_{ZM} = \psi_{eq}(I_1, I_2, I_3) + \psi_{neq}(I_1^s, I_2^s, I_3^s) = \frac{\lambda}{2} I_1^2 + \mu I_2 + \frac{\lambda^s}{2} (I_1^s)^2 + \mu^s I_2^s \quad (61b)$$

$$\rho_0 \psi_{PTM} = \psi_{eq}(I_1, I_2, I_3) + \psi_{neq}(I_1^k, I_2^k, I_3^k) = \frac{\lambda}{2} I_1^2 + \mu I_2 + \frac{\lambda^k}{2} (I_1^k)^2 + \mu^k I_2^k \quad (61c)$$

where $\lambda, \mu, \lambda^s, \mu^s, \lambda^k$ and μ^k are associated material parameters, determined from experiments. It is worth noting that the prediction capability of the developed models could be improved by including higher order terms in the Taylor series expansion or choosing different functional forms for the Helmholtz free energy (e.g., Anssari-Benam and Hossain, 2023). However, the primary goal of this paper is to develop physics-based constitutive models for Kelvin–Voigt type viscoelastic solids. Hence, the higher order terms have been avoided

to reduce the number of material parameters and keep the material models more tractable. Nevertheless the representative stress-strain curves show good match with the experimental data, as will be shown in the following section. Let us now explore the steps involved in the numerical solutions for these models. Here we assume that the test is displacement-controlled and thus, the history of the total deformation gradient $\mathbf{F}$ is known at all times. The integration algorithms for these models proceed as follows:

1. *Discretization of the total time interval:* Let us start with the discretization of the total time for the test $[0, T]$ into smaller intervals such that $\bigcup_{n=1}^k [t_n, t_{n+1}] = [0, T]$. Here h is a small time increment or a time step such that $t_{n+1} = t_n + h$ and k denotes the total number of time intervals. At the previous time step t_n , the deformation gradient $\mathbf{F}_n$ as well as all other variables such as $\mathbf{S}_n$, $\mathbf{E}_n^v$ etc. are known. Since the test is assumed to be displacement-controlled, the increment in the deformation gradient and thus, its value at the current time step, $\mathbf{F}_{n+1}$ is also known. The total Green strain $\mathbf{E}_{n+1}$ and its rate $\dot{\mathbf{E}}_{n+1}$ can be evaluated from the known values of the deformation gradient, using Eq. (3). For a Kelvin–Voigt model, the required rate of the Green strain can be computed as

$$\dot{\mathbf{E}}_{n+1} = \frac{\mathbf{E}_{n+1} - \mathbf{E}_n}{h} + \mathcal{O}(h^2). \quad (62)$$

2. *Numerical integration of the evolution equations:* With these information, now we proceed to a numerical integration of the evolution Eqs. (21) and (48) for the Maxwell/Zener and Poynting–Thompson model, respectively. Here we use a fourth-order exponential time-differencing Runge–Kutta (ETD4RK) method, developed by Cox and Matthews (2002). The use of exponential integrator is necessary for preserving the eigen-structure of the relevant internal variables. In this method, the internal variable $\mathbf{E}_{n+1}^v$ is determined with the help of its value at the previous time step as well as the values of $\mathbf{E}$ determined at different intermediate steps between $[t_n, t_{n+1}]$.

In order to update $\mathbf{E}_{n+1}^v$ using this method, the evolution equations for the Maxwell, Zener, and Poynting–Thomson models are first re-written in the form

$$\dot{\mathbf{E}}^v = -c \mathbf{E}^v + f(\mathbf{E}^v, \mathbf{E}(t)). \quad (63)$$

where c is a constant that depends on the material parameters and $f(\mathbf{E}^v, \mathbf{E}(t))$ is a nonlinear function of the internal variable $\mathbf{E}^v$ and the Green strain $\mathbf{E}(t)$. In view of Eq. (21), the expressions for c and $f(\mathbf{E}^v, \mathbf{E}(t))$ for the Maxwell and Zener models read

$$c = -2\mu/\eta \quad \text{and} \quad f(\mathbf{E}^v, \mathbf{E}(t)) = \frac{1}{\eta} \left[\frac{\lambda}{2} \text{tr}(\mathbf{E} - \mathbf{E}^v) \mathbf{I} + 2\mu \mathbf{E} \right] \quad (64)$$

whereas for the Poynting–Thomson model, these functions are evaluated from Eq. (48) as

$$c = -\frac{1}{\eta} (2\mu + 2\mu^k) \quad \text{and} \quad f(\mathbf{E}^v, \mathbf{E}(t)) = \frac{1}{\eta} \left[\frac{\lambda}{2} \text{tr}(\mathbf{E} - \mathbf{E}^v) \mathbf{I} + 2\mu \mathbf{E} - \lambda^k \text{tr}(\mathbf{E}^v) \right]. \quad (65)$$

Now to update $\mathbf{E}^v$, Eq. (63) is integrated over the time interval $[t_n, t_{n+1}]$ to obtain

$$\mathbf{E}^v(t_{n+1}) = \exp(c h) \mathbf{E}_n^v + \int_0^h \exp(c(h-s)) f(\mathbf{E}^v(t_n + s), \mathbf{E}(t_n + s)) ds. \quad (66)$$

Although Eq. (66) provides an exact solution to Eq. (63), computing the associated integral analytically for a general expression of the function $f(\mathbf{E}^v, \mathbf{E}(t))$ is a difficult task, if at all possible. Therefore, we need to construct a suitable numerical approximation to this integral using a fourth-order exponential time-differencing Runge–Kutta (ETD4RK) method (Cox and Matthews, 2002). In this method, we first calculate the values of three variables— a_n , b_n and c_n , that represent the internal variable evaluated at different intermediate time steps $t \in [t_n, t_{n+1}]$ and,

are later used for the numerical evaluation of $\mathbf{E}_{n+1}^v$. The expressions for a_n , b_n and c_n are given as

$$a_n = \mathbf{E}_n^v \exp(ch/2) + [(\exp(ch/2) - 1)/c] f(\mathbf{E}^v, \mathbf{E}(t_n)) \quad (67a)$$

$$b_n = \mathbf{E}_n^v \exp(ch/2) + [(\exp(ch/2) - 1)/c] f(a_n, \mathbf{E}(t_n + h/2)) \quad (67b)$$

$$c_n = a_n \exp(ch/2) + [(\exp(ch/2) - 1)/c] (2f(b_n, \mathbf{E}(t_n + h/2)) - f(\mathbf{E}^v, \mathbf{E}(t_n))) \quad (67c)$$

Note that the function f is considered to be an explicit function of both time t and the internal variable in the original formulation. However, in our framework, while f remains an explicit function of the internal variable $\mathbf{E}^v$, its dependence on time t becomes an implicit one through the Green strain, $\mathbf{E}(t)$. Thus, a_n , b_n and c_n are evaluated by substituting the Green strain $\mathbf{E}$ evaluated at the prescribed intermediate time steps. With the help of a_n , b_n and c_n , we finally evaluate $\mathbf{E}_{n+1}^v$ as

$$\begin{aligned} \mathbf{E}_{n+1}^v = & \mathbf{E}_n^v \exp(ch) + \{ f(\mathbf{E}^v, \mathbf{E}(t_n)) [-4 - hc + \exp(ch)(4 - 3hc + h^2c^2)] \\ & + 4f(a_n, \mathbf{E}(t_n + h/2)) [2 + hc + \exp(ch)(-2 + hc)] \\ & + f(\mathbf{E}^v, \mathbf{E}(t_n + h)) [-4 - 3hc - h^2c^2 + \exp(ch)(4 - hc)] \} / h^2 c^3. \end{aligned} \quad (68)$$

3. *Calculation of the stresses:* With the kinematic variables evaluated at the current time step t_{n+1} , the second Piola–Kirchhoff stress can now be easily calculated at this time step. For the chosen Helmholtz free energy in Eq. (61), the second Piola–Kirchhoff stress for a Maxwell and a Poynting–Thompson model is computed as

$$\begin{aligned} \mathbf{S}_{n+1} = & \alpha_1 \mathbf{I} + 2\alpha_2 (\mathbf{E}_{n+1} - \mathbf{E}_{n+1}^v) + 3\alpha_3 (\mathbf{E}_{n+1} - \mathbf{E}_{n+1}^v)^2 \\ & = (\lambda/2) \text{tr}(\mathbf{E}_{n+1} - \mathbf{E}_{n+1}^v) \mathbf{I} + 2\mu (\mathbf{E}_{n+1} - \mathbf{E}_{n+1}^v) \end{aligned} \quad (69)$$

whereas for a Zener model, the expression for the second Piola–Kirchhoff stress becomes

$$\begin{aligned} \mathbf{S}_{n+1} = & (\alpha_1 + \alpha_4) \mathbf{I} + 2(\alpha_2 + \alpha_5) \mathbf{E}_{n+1} - 2\alpha_2 \mathbf{E}_{n+1}^v + 3\alpha_3 (\mathbf{E}_{n+1} - \mathbf{E}_{n+1}^v)^2 + 3\alpha_6 \mathbf{E}_{n+1}^2 \\ & = (\lambda/2) \text{tr}(\mathbf{E}_{n+1} - \mathbf{E}_{n+1}^v) \mathbf{I} + (\lambda^s/2) \text{tr}(\mathbf{E}_{n+1}) \mathbf{I} + 2\mu (\mathbf{E}_{n+1} - \mathbf{E}_{n+1}^v) + 2\mu^s \mathbf{E}_{n+1}. \end{aligned} \quad (70)$$

The last of Eqs. (69) and (70) result in from the specific choice for the Helmholtz free energy, provided in Eq. (61). For a similar choice of Helmholtz free energy for a Kelvin–Voigt model, the second Piola Kirchhoff can be directly computed using steps 1 and 3 as

$$\mathbf{S}_{n+1} = \lambda^k \text{tr}(\mathbf{E}_{n+1}^v) \mathbf{I} + 2\mu^k \mathbf{E}_{n+1}^v + \eta \frac{(\mathbf{E}_{n+1} - \mathbf{E}_n)}{h}. \quad (71)$$

4.2. Application to a uniaxial stretch

We now consider a simple homogeneous deformation such as a uniaxial, isochoric stretch of a solid cylinder to demonstrate the response of our model as well as for experimental verification. For this purpose, let us write the deformation map in a cylindrical polar coordinate system. The motion of a material particle is given by

$$r = R, \quad \theta = \Theta, \quad z = \Lambda(t) Z \quad (72)$$

where $\Lambda(t)$ is a total uniaxial stretch applied to a sample. Since the motion is assumed to be isochoric, the deformation gradient $\mathbf{F}$ is calculated as

$$\mathbf{F}(t) = \text{diag}(1/\sqrt{\Lambda(t)}, 1/\sqrt{\Lambda(t)}, \Lambda(t)) \quad (73)$$

For this motion, the right Cauchy–Green tensor $\mathbf{C}$ and the Green–Lagrange strain tensor $\mathbf{E}$ are obtained as

$$\mathbf{C} = \text{diag}\left(\frac{1}{\Lambda}, \frac{1}{\Lambda}, \Lambda^2\right), \quad \mathbf{E} = \text{diag}\left(\frac{1-\Lambda}{2\Lambda}, \frac{1-\Lambda}{2\Lambda}, \frac{\Lambda^2-1}{2}\right). \quad (74)$$

Now that the history of deformation is known, we can use the numerical algorithms developed in Section 4.1 to illustrate the response of the developed models for different functional forms of history of stretch, $\Lambda(t)$.

Table 1
Material parameters used in the numerical examples.

	λ	μ	λ^k	μ^k	η
Material-A	500	300	200	100	5000
VHB 4910 (Hossain et al., 2012; Wang et al., 2016)	6408	100	0.8	0.32	345.3
PVB (Hooper et al., 2012)	50	8	1	0.5	0.003

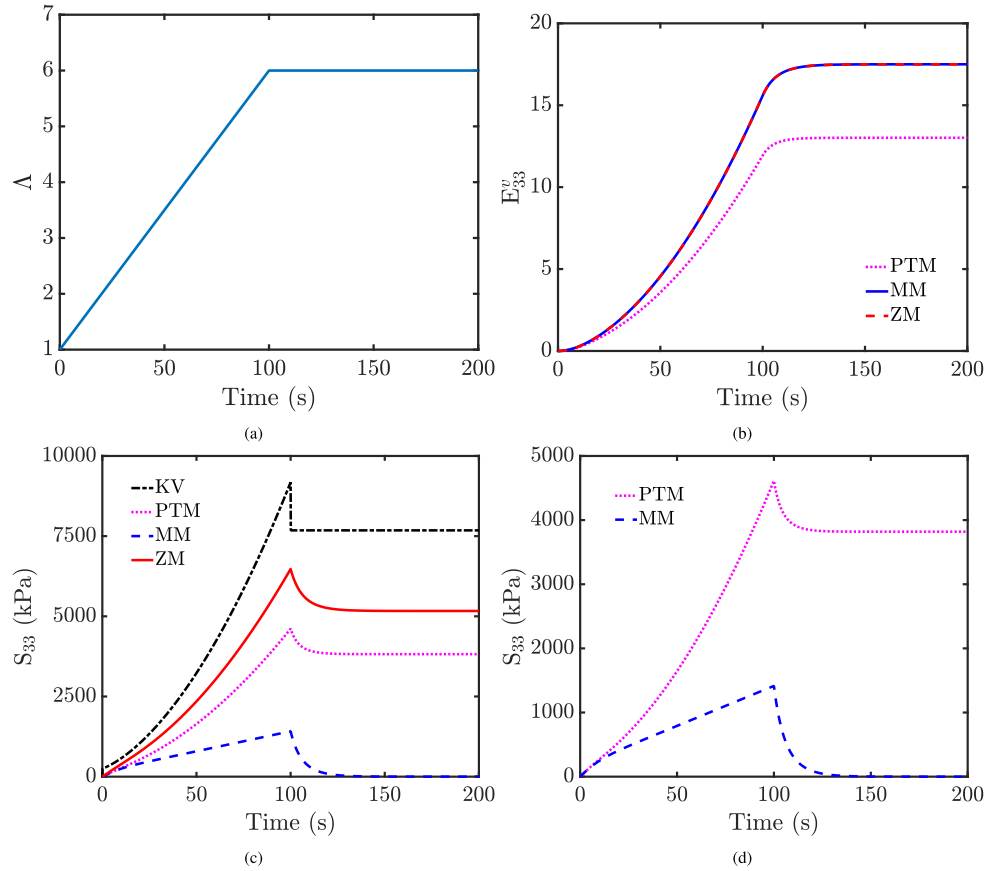


Fig. 3. A representative response of the developed models: (a) Applied stretch with respect to time, (b) The evolution of the viscous strain for the Maxwell, Zener and Poynting–Thompson models, (c) The stress response and relaxation behavior of the developed models, (d) An enlarged version of the stress response of the Poynting–Thompson and the Maxwell models.

4.2.1. Response of the developed models

To investigate the stress relaxation behavior and the evolution of the viscous strain, E^v of the developed models, we apply a linear stretch of the form $\Lambda(t) = 1 + \dot{\Lambda} t$ with $\dot{\Lambda} = 0.05 \text{ s}^{-1}$ for a time of 100 s followed by a constant stretch of $\Lambda(t) = 6$ as shown in Fig. 3(a). The stress responses of all four developed models are presented together in Fig. 3(c), with the enlarged versions of the Poynting–Thompson model (PTM) and the Maxwell Model (MM) presented in Fig. 3(d). The material parameters used for this numerical example are given in Table 1 (material-A). Fig. 3(b) shows the evolution of the viscous strain over time for the Poynting–Thompson (PTM), Maxwell (MM) and the Zener model (ZM). As expected, one can observe a change in the slope of this curve in all three models around $t = 100 \text{ s}$ when the stretch rate reduces to zero. It is observed that the evolution of the viscous strain for the Maxwell and the Zener models coincide with each other. This is due to the fact that the evolution equations for the Maxwell and Zener model are the same as shown in Section 2.3.

For the stress response (Fig. 3(c)), all four material models show a steady increase in stress for the first 100 s when the stretch Λ is linearly increasing with time. During this time, all four material models show a

typical constitutive behavior as observed in soft viscoelastic materials undergoing large deformations. At the initial stage of deformation, all four models show a steep increase in stress for a short duration of time, followed by a more gradual increase. This behavior can be more prominently observed in Fig. 3(d). It is worth noting that the exact shape of the stress–strain curve depends on the material parameters as well as the stretch rate under consideration.

After $t = 100 \text{ s}$, the stretch is held constant at $\Lambda = 6$. At this stage, a stress relaxation can be prominently observed in the Maxwell, Zener, and the Poynting–Thompson model. Quite evidently, the Kelvin–Voigt model does not offer any noticeable relaxation and shows an elastic response under a zero stretch rate instead. It can be observed that while the Maxwell model eventually relaxes to an almost vanishing stress, the other solid models relax to a constant value. Moreover, the Zener model shows a stress relaxation that eventually aligns with the response of the (relaxed) Kelvin–Voigt model. This is due to the fact that the Kelvin–Voigt model is derived a limiting case from the Zener model in Section 3.3.2. Since the derived Zener and Poynting–Thompson models exhibit qualitatively similar responses as shown in Fig. 3, it is possible to fit both these model predictions with experimental data when

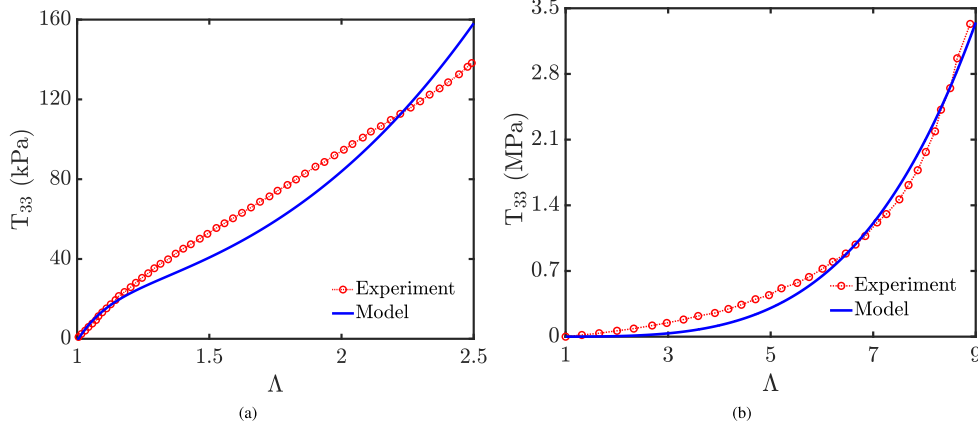


Fig. 4. Experimental verification of the Poynting–Thompson model in terms of the Cauchy stress $\mathbf{T}$ and applied stretch Λ : comparison with the experimental results of (a) Hossain et al. (2012) up to a total stretch of $\Lambda = 2.5$ and a stretch rate of 0.05 s^{-1} and (b) Wang et al. (2016) up to a large stretch of $\Lambda = 9$ and a stretch rate of $9 \times 10^{-5} \text{ s}^{-1}$.

appropriate material parameters are selected. However, the underlying physical mechanisms for both these models remain distinct, which will be further elucidated in Section 5.

4.2.2. Experimental verification of the Poynting–Thompson model

Since the Zener and Maxwell model have long been explored, the developed Poynting–Thompson model is our primary model of interest. First, we verify this model with experimental results obtained from the uniaxial tension tests of polymers. Hossain et al. (2012) and Wang et al. (2016) performed a uniaxial tension test on VHB 4910 polymer at stretch rates of different orders of magnitude. Since both these experiments used the same material, we also keep the material parameters the same for our numerical modeling. These material parameters are presented in Table 1 (VHB 4910). Here we use a linear stretch of $\Lambda(t) = 1 + \dot{\Lambda}t$ with the stretch rates same as the ones used in the corresponding experiments. Here we use the experimental observation of Hossain et al. (2012) at a stretch rate of 0.05 s^{-1} up to a total stretch of 2.5 and, the results of Wang et al. (2016) with a very small stretch rate of $0.9 \times 10^{-4} \text{ s}^{-1}$ up to a very large stretch of $\Lambda = 9$. Comparison of the experimental results with our model prediction is shown in Fig. 4. The model predictions for both stretch rates show a very good overall match with the experimental data. This is suggested by the R^2 -values in Figs. 4(a) and 4(b) which are reported as 0.94 and 0.98, respectively.

Although our model is able to capture the overall constitutive behavior of VHB 4910 for a wide range of stretch, as shown in Fig. 4, the fitting with the experimental data at each point could be improved by considering higher-order terms in Eq. (61). This exercise, however, is currently beyond the scope of this paper.

4.2.3. Response of the Poynting–Thompson model at different stretch rates

We now seek to understand the stress response of the Poynting–Thompson solid under various stretch rates. For this purpose, we select a wide range of stretch rates of 0.2 s^{-1} , 2 s^{-1} , 20 s^{-1} and 400 s^{-1} . The model predictions are compared with the experimental results of Hooper et al. (2012), obtained from a uniaxial tensile test on poly (vinyl butyral) (PVB) polymers. For this comparison, we have used the material parameters listed in Table 1 (PVB). Comparison between the experimental results and model predictions are shown in Fig. 5. The results show that for a stretch of $\Lambda = 3$, the model shows an excellent match for all the stretch rates under consideration. The R^2 values for these comparisons are provided in Table 2.

5. Discussions on anisotropic Poynting–Thompson solids

In this section, we extend the derived Poynting–Thompson model into other material symmetry groups. Specifically, a transversely

isotropic, orthotropic and monoclinic Poynting–Thompson model have been derived.

5.1. Transversely isotropic materials

A transversely isotropic material is characterized by a single preferred material direction at each point $\mathbf{X} \in \kappa_r(B)$. Let this direction be defined by a unit vector $\mathbf{n}(\mathbf{X})$, which is perpendicular to the plane of isotropy. The material response is isotropic within this plane and anisotropic along $\mathbf{n}$. Accordingly, the constitutive behavior depends on the structural tensor $\mathbf{N} = \mathbf{n} \otimes \mathbf{n}$ (Spencer, 1982; Taylor et al., 2009; Balbi et al., 2018; Sadik and Yavari, 2024). For this material, the Helmholtz free energy can be written as

$$\psi = \bar{\psi}(I_1, I_2, I_3, I_4, I_5, I_1^k, I_2^k, I_3^k, I_4^k, I_5^k) \quad (75)$$

Here the additional invariants are defined as

$$I_4 = \mathbf{n} \cdot \bar{\mathbf{E}}^e \mathbf{n}, \quad I_5 = \mathbf{n} \cdot \bar{\mathbf{E}}^{e^2} \mathbf{n}, \quad I_4^k = \mathbf{n} \cdot \mathbf{E}^v \mathbf{n}, \quad \text{and} \quad I_5^k = \mathbf{n} \cdot \mathbf{E}^{v^2} \mathbf{n}. \quad (76)$$

Using Eqs. (11) and (36), the resulting second Piola–Kirchhoff stress and the configurational force are evaluated as

$$\begin{aligned} \tilde{\mathbf{S}} &= \alpha_1 \mathbf{I} + 2\alpha_2 (\mathbf{E} - \mathbf{E}^v) \quad \text{where} \\ \tilde{\mathbf{S}} &= \mathbf{S} - 3\alpha_3 (\mathbf{E} - \mathbf{E}^v)^2 - \alpha_7 \mathbf{N} - 2\alpha_8 [(\mathbf{E} - \mathbf{E}^v) \mathbf{N} + \mathbf{N} (\mathbf{E} - \mathbf{E}^v)] \end{aligned} \quad (77)$$

and,

$$\mathbf{A}^v = -\rho_0 \frac{\partial \psi}{\partial \mathbf{E}^v} = \mathbf{S} - \alpha_4 \mathbf{I} - 2\alpha_5 \mathbf{E}^v - 3\alpha_6 \mathbf{E}^{v^2} - \alpha_9 \mathbf{N} - 2\alpha_{10} (\mathbf{E}^v \mathbf{N} + \mathbf{N} \mathbf{E}^v). \quad (78)$$

Since $\mathbf{A}^v$ acts as the driving force for the evolution of the natural configuration, we choose the rate of dissipation function as

$$\xi = \frac{1}{\eta_1} \mathbf{A}^v : \mathbf{A}^v + \frac{1}{\eta_2} (\mathbf{N} \mathbf{A}^v : \mathbf{N} \mathbf{A}^v) \quad (79)$$

The second term in the expression of ξ takes into account the effect of anisotropy. Since the chosen rate of dissipation, ξ is a quadratic function in $\mathbf{A}^v$ and $\mathbf{N} \mathbf{A}^v$ with the viscosity parameter $\eta_1, \eta_2 > 0$, it can easily be verified that the rate of dissipation function ξ is convex in the configurational force space. Now using Eqs. (40) and (79) on the chosen rate of dissipation function, one can obtain the evolution equation for the viscous strain as

$$\dot{\mathbf{E}}^v = \mathbf{G}_t \mathbf{A}^v \quad \text{where} \quad \mathbf{G}_t = \frac{1}{\eta_1} \mathbf{I} + \frac{1}{\eta_2} \mathbf{N}^\top \mathbf{N}. \quad (80)$$

Note that the second-order tensor $\mathbf{G}_t$ is associated with the viscosity parameters η_1 and η_2 as well as the structural tensor $\mathbf{N}$.

Now taking a material time derivative of Eq. (77) and using Eqs. (7) and (80), the resulting governing equation is derived as

$$\dot{\tilde{\mathbf{S}}} + 2\alpha_2 \mathbf{G}_t \mathbf{S}$$

Table 2 R^2 values for the matches between model prediction and experimental data in Figs. 4 and 5.

		VHB 4910 (Hossain et al., 2012; Wang et al., 2016)		
Stretch rate	0.05 s ⁻¹	9 × 10 ⁻⁵ s ⁻¹		
R ²	0.95	0.98		
		Poly (vinyl butyral) (Hooper et al., 2012)		
Stretch rate	400 s ⁻¹	20 s ⁻¹	2 s ⁻¹	0.2 s ⁻¹
R ²	0.99	0.99	0.99	0.99

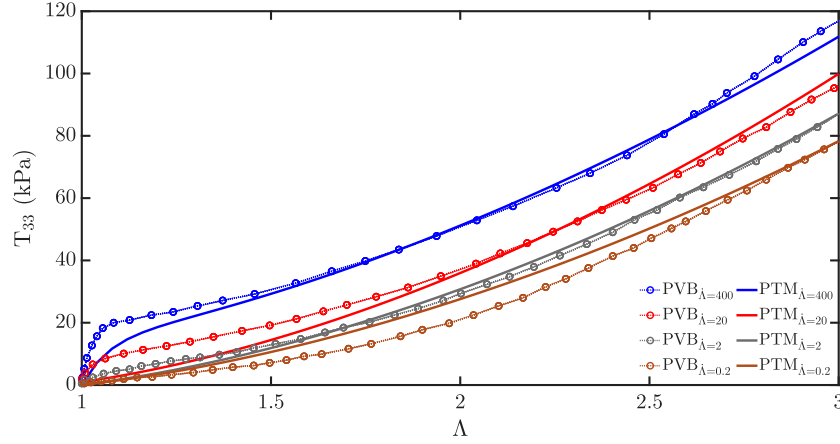


Fig. 5. Response of the Poynting–Thompson model at different stretch rates. The response of the model in terms of the Cauchy stress and the applied stretch at a wide range of stretch rates of 400 s⁻¹, 20 s⁻¹, 2 s⁻¹ and 0.2 s⁻¹ is compared with the experimental data of Hooper et al. (2012). The model predictions show an excellent match with the experimental results.

$$= 2\alpha_2 \dot{\mathbf{E}} + 2\alpha_2 \mathbf{G}_t \left[\alpha_4 \mathbf{I} + 2\alpha_5 \mathbf{E}^v + 3\alpha_6 \mathbf{E}^{v^2} + \alpha_9 \mathbf{N} + 2\alpha_{10} (\mathbf{E}^v \mathbf{N} + \mathbf{N} \mathbf{E}^v) \right]. \quad (81)$$

From Eqs. (49) and (88), it is worth noting that when the structural tensor $\mathbf{N} = \mathbf{I}$, the isotropic Poynting–Thompson solid is retrieved with slight modifications in the associated coefficients.

5.2. Orthotropic materials

An orthotropic solid is characterized by three mutually orthogonal preferred material directions at each point $\mathbf{X} \in \kappa_r(\mathcal{B})$, denoted by $\mathbf{n}_1(\mathbf{X})$, $\mathbf{n}_2(\mathbf{X})$, and $\mathbf{n}_3(\mathbf{X})$. Their associated structural tensors are defined as $\mathbf{N}_i = \mathbf{n}_i \otimes \mathbf{n}_i$ $i = 1, 2, 3$, of which only two are independent due to the relation $\mathbf{N}_1 + \mathbf{N}_2 + \mathbf{N}_3 = \mathbf{I}$ (Spencer, 1982; Sadik and Yavari, 2024). Accordingly, the Helmholtz free energy function is assumed to have the following form:

$$\psi = \bar{\psi}(I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_1^k, I_2^k, I_3^k, I_4^k, I_5^k, I_6^k, I_7^k) \quad (82)$$

In addition to the previously used invariants, the following additional invariants are introduced:

$$\begin{aligned} I_4 &= \mathbf{n}_1 \cdot \bar{\mathbf{E}}^e \mathbf{n}_1, & I_5 &= \mathbf{n}_1 \cdot \bar{\mathbf{E}}^{e^2} \mathbf{n}_1, & I_6 &= \mathbf{n}_2 \cdot \bar{\mathbf{E}}^e \mathbf{n}_2, & I_7 &= \mathbf{n}_2 \cdot \bar{\mathbf{E}}^{e^2} \mathbf{n}_2, \\ I_4^k &= \mathbf{n}_1 \cdot \mathbf{E}^v \mathbf{n}_1, & I_5^k &= \mathbf{n}_1 \cdot \mathbf{E}^{v^2} \mathbf{n}_1, & I_6^k &= \mathbf{n}_2 \cdot \mathbf{E}^v \mathbf{n}_2, & \text{and, } & I_7^k &= \mathbf{n}_2 \cdot \mathbf{E}^{v^2} \mathbf{n}_2. \end{aligned} \quad (83)$$

The second Piola–Kirchhoff stress and the configurational force are obtained by using Eqs. (11) and (36) as

$$\begin{aligned} \bar{\mathbf{S}} &= \alpha_1 \mathbf{I} + 2\alpha_2 (\mathbf{E} - \mathbf{E}^v) \\ \text{with } \bar{\mathbf{S}} &= \mathbf{S} - 3\alpha_3 (\mathbf{E} - \mathbf{E}^v)^2 - \alpha_7 \mathbf{N}_1 - 2\alpha_8 \{(\mathbf{E} - \mathbf{E}^v) \mathbf{N}_1 + \mathbf{N}_1 (\mathbf{E} - \mathbf{E}^v)\} - \alpha_{11} \mathbf{N}_2 \\ &\quad - 2\alpha_{12} \{(\mathbf{E} - \mathbf{E}^v) \mathbf{N}_2 + \mathbf{N}_2 (\mathbf{E} - \mathbf{E}^v)\} \end{aligned} \quad (84)$$

and,

$$\begin{aligned} \mathbf{A}^v &= \mathbf{S} - \alpha_4 \mathbf{I} - 2\alpha_5 \mathbf{E}^v - 3\alpha_6 \mathbf{E}^{v^2} - \alpha_9 \mathbf{N}_1 - 2\alpha_{10} (\mathbf{E}^v \mathbf{N}_1 + \mathbf{N}_1 \mathbf{E}^v) \\ &\quad - \alpha_{13} \mathbf{N}_2 - 2\alpha_{14} (\mathbf{E}^v \mathbf{N}_2 + \mathbf{N}_2 \mathbf{E}^v). \end{aligned} \quad (85)$$

Here we choose the rate of dissipation function as a quadratic function of the configurational force $\mathbf{A}^v$ as well as the associated structural tensors with the viscosity parameters $\eta_1, \eta_2, \eta_3 > 0$. The form for the rate of dissipation function is chosen as

$$\xi = \frac{1}{\eta_1} \mathbf{A}^v : \mathbf{A}^v + \frac{1}{\eta_2} (\mathbf{N}_1 \mathbf{A}^v : \mathbf{N}_1 \mathbf{A}^v) + \frac{1}{\eta_3} (\mathbf{N}_2 \mathbf{A}^v : \mathbf{N}_2 \mathbf{A}^v). \quad (86)$$

Now using Eqs. (40) and (79), one can obtain the evolution equation for the viscous strain as

$$\dot{\mathbf{E}}^v = \mathbf{G}_o \mathbf{A}^v \quad \text{where } \mathbf{G}_o = \frac{1}{\eta_1} \mathbf{I} + \frac{1}{\eta_2} \mathbf{N}_1^\top \mathbf{N}_1 + \frac{1}{\eta_3} \mathbf{N}_2^\top \mathbf{N}_2. \quad (87)$$

Now taking a material time derivative of Eq. (77) and using Eqs. (7) and (80), we obtain the governing equation as

$$\begin{aligned} \dot{\bar{\mathbf{S}}} + 2\alpha_2 \mathbf{G}_o \mathbf{S} &= 2\alpha_2 \dot{\mathbf{E}} + 2\alpha_2 \mathbf{G}_o \left\{ \alpha_4 \mathbf{I} + 2\alpha_5 \mathbf{E}^v + 3\alpha_6 \mathbf{E}^{v^2} + \alpha_9 \mathbf{N}_1 \right. \\ &\quad \left. + 2\alpha_{10} (\mathbf{E}^v \mathbf{N}_1 + \mathbf{N}_1 \mathbf{E}^v) + \alpha_{13} \mathbf{N}_2 + 2\alpha_{14} (\mathbf{E}^v \mathbf{N}_2 + \mathbf{N}_2 \mathbf{E}^v) \right\}. \end{aligned} \quad (88)$$

5.3. Monoclinic materials

For a monoclinic solid, the preferred material directions $\mathbf{n}_1$ and $\mathbf{n}_2$ are not orthogonal, i.e., $\mathbf{n}_1 \cdot \mathbf{n}_2 \neq 0$, while $\mathbf{n}_3$ is perpendicular to the plane spanned by $\mathbf{n}_1$ and $\mathbf{n}_2$. The structural tensors $\mathbf{N}_1 = (\mathbf{n}_1 \otimes \mathbf{n}_1)$, and $\mathbf{N}_2 = (\mathbf{n}_2 \otimes \mathbf{n}_2)$, describe the directional effects along the material axis while $\mathbf{M} = (\mathbf{n}_1 \otimes \mathbf{n}_2) + (\mathbf{n}_2 \otimes \mathbf{n}_1)$ account for coupling between these direction. For monoclinic materials, we assume the form for ψ as

$$\psi = \bar{\psi}(I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8, I_9, I_1^k, I_2^k, I_3^k, I_4^k, I_5^k, I_6^k, I_7^k, I_8^k, I_9^k) \quad (89)$$

Following the previous formulation, we take the same set of invariants along with four additional invariants, given as

$$I_8 = \mathbf{M} : \tilde{\mathbf{E}}^e, \quad I_9 = \mathbf{M} : \tilde{\mathbf{E}}^{e^2}, \quad I_8^k = \mathbf{M} : \mathbf{E}^v \quad \text{and}, \quad I_9^k = \mathbf{M} : \mathbf{E}^{v^2}. \quad (90)$$

The second Piola–Kirchhoff stress and the configurational force are derived as

$$\begin{aligned} \tilde{\mathbf{S}} &= \alpha_1 \mathbf{I} + 2\alpha_2 (\mathbf{E} - \mathbf{E}^v) \\ \text{with } \tilde{\mathbf{S}} &= \mathbf{S} - 3\alpha_3 (\mathbf{E} - \mathbf{E}^v)^2 - \alpha_7 \mathbf{n}_1 - 2\alpha_8 \{(\mathbf{E} - \mathbf{E}^v) \mathbf{N}_1 + \mathbf{N}_1 (\mathbf{E} - \mathbf{E}^v)\} - \alpha_{11} \mathbf{N}_2 \\ &\quad - 2\alpha_{12} \{(\mathbf{E} - \mathbf{E}^v) \mathbf{N}_2 + \mathbf{N}_2 (\mathbf{E} - \mathbf{E}^v)\} - \alpha_{15} \mathbf{M} \\ &\quad - 2\alpha_{16} \{(\mathbf{E} - \mathbf{E}^v) \mathbf{M} + \mathbf{M} (\mathbf{E} - \mathbf{E}^v)\} \end{aligned} \quad (91)$$

and,

$$\begin{aligned} \mathbf{A}^v &= -\rho_0 \frac{\partial \psi}{\partial \mathbf{E}^v} = \mathbf{S} - \alpha_4 \mathbf{I} - 2\alpha_5 \mathbf{E}^v - 3\alpha_6 \mathbf{E}^{v^2} - \alpha_9 \mathbf{N}_1 - 2\alpha_{10} (\mathbf{E}^v \mathbf{N}_1 + \mathbf{N}_1 \mathbf{E}^v) \\ &\quad - \alpha_{13} \mathbf{N}_2 - 2\alpha_{14} (\mathbf{E}^v \mathbf{N}_2 + \mathbf{N}_2 \mathbf{E}^v) - \alpha_{17} \mathbf{M} - 2\alpha_{18} (\mathbf{E}^v \mathbf{M} + \mathbf{M} \mathbf{E}^v). \end{aligned} \quad (92)$$

Now we have chosen a rate of dissipation that is a quadratic function of the configurational force $\mathbf{A}^v$ with the viscosity parameters $\eta_1, \eta_2, \eta_3, \eta_4 > 0$,

$$\xi = \frac{1}{\eta_1} \mathbf{A}^v : \mathbf{A}^v + \frac{1}{\eta_2} (\mathbf{N}_1 \mathbf{A}^v : \mathbf{N}_1 \mathbf{A}^v) + \frac{1}{\eta_3} (\mathbf{N}_2 \mathbf{A}^v : \mathbf{N}_2 \mathbf{A}^v) + \frac{1}{\eta_4} (\mathbf{N}_1 \mathbf{A}^v : \mathbf{N}_2 \mathbf{A}^v) \quad (93)$$

Now using Eqs. (40) and (79) on the chosen rate of dissipation function, one can obtain the evolution equation for the viscous strain.

$$\dot{\mathbf{E}}^v = \mathbf{G}_m : \mathbf{A}^v \quad \text{where } \mathbf{G}_m = \frac{1}{\eta_1} \mathbf{I} + \frac{1}{\eta_2} \mathbf{N}_1^\top \mathbf{N}_1 + \frac{1}{\eta_3} \mathbf{N}_2^\top \mathbf{N}_2 + \frac{1}{2\eta_4} (\mathbf{N}_1^\top \mathbf{N}_2 + \mathbf{N}_2^\top \mathbf{N}_1). \quad (94)$$

Now taking a material time derivative of Eq. (77) and using Eqs. (7) and (80), the governing equation is obtained as

$$\begin{aligned} \dot{\tilde{\mathbf{S}}} &+ 2\alpha_2 \mathbf{G}_m \mathbf{S} \\ &= 2\alpha_2 \dot{\tilde{\mathbf{E}}} + 2\alpha_2 \mathbf{G}_m \left\{ \alpha_4 \mathbf{I} + 2\alpha_5 \mathbf{E}^v + 3\alpha_6 \mathbf{E}^{v^2} + \alpha_9 \mathbf{N}_1 + 2\alpha_{10} (\mathbf{E}^v \mathbf{N}_1 + \mathbf{N}_1 \mathbf{E}^v) \right. \\ &\quad \left. + \alpha_{13} \mathbf{N}_2 + 2\alpha_{14} (\mathbf{E}^v \mathbf{N}_2 + \mathbf{N}_2 \mathbf{E}^v) + \alpha_{17} \mathbf{M} + 2\alpha_{18} (\mathbf{E}^v \mathbf{M} + \mathbf{M} \mathbf{E}^v) \right\}. \end{aligned} \quad (95)$$

6. Concluding remarks

In this paper, we developed the nonlinear Kelvin–Voigt type viscoelastic models using the stress-space formulation of the theory of evolving natural configuration. These models are developed by appropriately choosing the Helmholtz free energy and considering a rate of dissipation as a function of the resulting configurational force according to the postulates of Singh and Paul (2025). We further show that a Maxwell and a Kelvin–Voigt model can be obtained as limiting cases from the developed Poynting–Thompson and Zener models, respectively. The integration algorithms for these models are developed and their efficacy has been shown through the numerical solution to the problem of a uniaxial stretch of polymers. The response of all four developed models for this problem has been discussed. The Poynting–Thompson model has been verified with experimental data and its response under different stretch rates has been demonstrated. Extension of the presented Poynting–Thompson model to other material symmetry groups, specifically a transversely isotropic, orthotropic and monoclinic solid has been discussed. For future direction, a full-scale finite element analysis can be developed for the proposed models.

CRediT authorship contribution statement

Tarun Singh: Writing – original draft, Visualization, Validation, Software, Methodology, Investigation, Formal analysis, Data curation. **Sandipan Paul:** Writing – review & editing, Supervision, Resources, Project administration, Methodology, Funding acquisition, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

The partial financial support for this research by the IIT Roorkee faculty initiation grant (# FIG-100976) is gratefully acknowledged.

Data availability

Data will be made available on request.

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