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A constitutive model for elastic–plastic materials using scalar conjugate stress/strain base pairs

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ABSTRACT

A constitutive model for elastic–plastic materials is developed using scalar, conjugate, stress–strain, base pairs in a finite deformation setting. These conjugate base pairs arise from an alternative **QR** decomposition of the deformation gradient that decomposes its matrix into an orthogonal rotation and an upper-triangular matrix, called the Laplace stretch. This decomposition is particularly useful from an experimental standpoint, as it enables one to directly measure the components of Laplace stretch and its plastic contributions in a specific coordinate system. Moreover, from an experimental standpoint, it is difficult to parameterize current material models due to a covariance between their tensor invariants, traditionally used in their constructions. The use of scalar, conjugate, base pairs are also helpful from that point of view. Interestingly, the multiplicative elastic–plastic decomposition of Laplace stretch leads to an additive decomposition of the total strain attributes into their corresponding elastic and plastic components. Although an additive strain decomposition is commonly used in small-strain theory, here such a decomposition is possible even for finite deformations. An additive decomposition of the strain attributes has a deeper consequence in the construction of our constitutive model. A maximum rate of dissipation criterion has been used in deriving our constitutive equations, as this criterion is valid for a wider class of materials. Two constitutive assumptions – one for a Helmholtz potential, and one for the rate of dissipation function – are required for our constitutive construction. This model does not presuppose the existence of a yield surface. In fact, it is shown that whether a material exhibits a yielding or a creep-like behavior depends upon the differentiability of the rate of dissipation function. Two cases of plastic deformation – volume-preserving and dilatant-pressure dependent deformations – have been considered. To illustrate the proposed model, finite strain versions of classical J_2 plasticity and a Drucker–Prager model are derived.

1. Introduction

In recent years, an alternative **QR** decomposition of a deformation gradient has been proposed that has several advantages over the traditional polar decomposition. In this decomposition, the matrix of the deformation gradient is decomposed into an orthogonal rotation matrix $\boldsymbol{\mathcal{R}}$ and an upper-triangular matrix $\boldsymbol{\mathcal{U}}$, called the Laplace stretch (Freed et al., 2019). Srinivasa (2012) showed that this decomposition is particularly useful with regards to interpreting experimental data, as components of the Laplace stretch carry physical meaning in a specific set of bases obtained by the Gram–Schmidt procedure. This set of bases, thus, is termed

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as an experimenter's or physical frame of reference. **QR** kinematics have been further explored by Freed and Srinivasa (2015), Lembo (2017), Freed and Zamani (2018), Paul and Freed (2020b) and Clayton (2020). Ghosh and Srinivasa (2014) extended this decomposition to elastoplasticity in their study of shape memory alloys. In this work, a multiplicative decomposition Bilby et al. (1957), Kröner (1959), Lee (1969) was first applied to the deformation gradient ($\mathbf{F} = \mathbf{F}^e\mathbf{F}^p$) and then a **QR** decomposition was applied to its plastic component. Thus, the elastic component of the deformation gradient remains a full matrix resulting in $\mathbf{F} = \mathbf{F}^e\boldsymbol{\mathcal{R}}^p\boldsymbol{\mathcal{U}}^p$. A different approach was adopted by Freed et al. (2019) where a **QR** decomposition was applied to the deformation gradient first, and the resulting Laplace stretch was decomposed into an elastic component $\boldsymbol{\mathcal{U}}^e$ and a plastic component $\boldsymbol{\mathcal{U}}^p$. Both $\boldsymbol{\mathcal{U}}^e$ and $\boldsymbol{\mathcal{U}}^p$ are upper-triangular matrices. The latter decomposition is possible due to the fact that the set of all invertible upper-triangular matrices form a group (thus, closed) under multiplication. Moreover, this property renders the elastic–plastic decomposition of Laplace stretch *unique*. This decomposition was used by Paul and Freed (2020a) to study the geometrically necessary dislocations.

Traditionally, tensor invariants are used in the construction of constitutive models whenever a polar decomposition of the deformation gradient is used. Despite the elegance of this theory, it suffers from a serious drawback as a covariance between tensor invariants hinders one's ability to parameterize a material model from an experimental standpoint (Criscione, 2004). In an attempt to avoid this issue, Freed et al. (2016) developed a constitutive theory that uses scalar, conjugate, stress/strain, base pairs. The physical meaning of the elements of Laplace stretch plays a crucial role in developing this constitutive theory. Although initially developed for characterization of 2-D biological membranes, this model was further extended to three-dimensional isotropic materials by Freed (2017) and anisotropic materials by Erel et al. (2019). Rajagopal and Srinivasa (2016) employed this kinematics to construct an implicit constitutive model for three-dimensional elastic bodies. Based on their kinematics, described in a locally convected coordinate system, Freed and Zamani (2019) developed constitutive relations for elastic bodies that take into account Kelvin, Poisson and Poynting effects.

In this paper, we develop a constitutive model for elastic–plastic materials using the scalar, conjugate, stress/strain, base pairs associated with **QR** kinematics. Development of constitutive models for elastic–plastic materials has been the central theme of many researchers' works (see Green and Naghdi, 1964, 1971; Rice, 1971; Hill and Rice, 1972, 1973; Naghdi and Trapp, 1975; Nemat-Nasser, 1982; Lubliner, 1984; Simo and Ortiz, 1985; Dafalias, 1987; Eve et al., 1990; Lubarda, 1991 and Miehe, 1998 for a partial list of references) as well as a longstanding point of contention (Casey and Naghdi, 1981; Naghdi, 1990) in the mechanics community for at least the past century. The property that distinguishes an elastic material from an inelastic one is the ability of the latter to dissipate energy, i.e., convert mechanical work done into heat. Therefore, thermodynamics play a crucial role in the development of constitutive models for inelastic materials. A standard thermodynamical approach based upon the Clausius–Duhem inequality provides rather weak guidance for the development of evolution equations for plastic strain and its rate. Therefore, several additional principles, e.g., maximum plastic work, maximum plastic dissipation, and Drucker's stability postulate have been used in the literature (Naghdi and Trapp, 1975; Lubliner, 1984; Simo and Hughes, 2006). In this work, we adopt the techniques of Rajagopal and Srinivasa (1998a,b) where it is considered that a body may possess multiple natural configurations. The response of a body is, thus, described as a family of elastic responses from these natural configurations. In this theory, two constitutive assumptions are made: one for the Helmholtz potential function ψ, and another for the dissipation function ξ. The evolution equation for the plastic strain rate is obtained by applying the principle of maximum rate of entropy production Rajagopal and Srinivasa (2004). Rajagopal and Srinivasa (2004) showed that although this principle is not followed by all materials, because it is not as fundamental as the second law of thermodynamics, it can be useful for a wide class of materials. This principle is also in sync with Onsager's principle of minimum rate of entropy production (1931) Onsager (1931) and is a generalized version of Ziegler's normality rule (1963) (Ziegler, 1963).

The paper is organized as follows. In Section 2, the **QR** kinematics is discussed in detail. Based upon this kinematics, we derive our basic constitutive equations in Section 3. First, the model is developed for a general case without any chosen form for the Helmholtz potential function. In Section 3.5, it is shown that with an assumed form for the Helmholtz potential function, the constitutive relations become significantly simple. Up to this point, the model is developed assuming that plastic deformation depends upon the dilatant-pressure, which is prevalent in some materials like soils, foams, rocks, etc. In Section 4, the case of volume-preserving plastic deformation is considered, which is commonly observed in metal plasticity. In Section 5, two example problems are derived in our framework: finite strain versions of the classical J_2 plasticity model, and a Drucker–Prager model. Finally, the key results are summarized and the paper is drawn to conclusion.

2. QR kinematics

Let us consider a body $\mathcal{B}$ with a typical material particle $\mathcal{P}$ in it. The position of particle $\mathcal{P}$ is defined by a vector $\boldsymbol{X}$ in a reference configuration of the body $\kappa_r(\mathcal{B})$ and by $\boldsymbol{x}$ in its current configuration $\kappa_t(\mathcal{B})$, respectively. The motion of a body is defined as $\boldsymbol{x}(\boldsymbol{X}, t) = \mathcal{X}(\boldsymbol{X}, t)$. The deformation gradient $\mathbf{F}$ takes a tangent vector from the reference configuration of a body and places it into a tangent space at its current configuration. The deformation gradient is defined as

$$\mathbf{F} = \frac{\partial \mathcal{X}(\boldsymbol{X}, t)}{\partial \boldsymbol{X}}. \tag{1}$$

We apply the Gram–Schmidt procedure to the matrix of a deformation gradient that decomposes it into an orthogonal rotation matrix, $\boldsymbol{\mathcal{R}}$, and an upper-triangular Laplace stretch, $\boldsymbol{\mathcal{U}}$. In this procedure, a new set of bases is obtained by Laplace's technique of successive orthogonal projections, with a particular coordinate direction and its adjoining coordinate plane being specified. It is in this new set of bases where the matrix of the deformation gradient takes on the form of an upper-triangular matrix. Because the coordinate direction and coordinate plane, knowledge required by the Gram–Schmidt procedure (McLellan, 1980), are not known *a priori*, a strategy to obtain this new set of bases, dependent upon the deformation under consideration, has been suggested by Paul

et al. (2021). This strategy re-indexes the matrix of a deformation gradient to remove any ambiguity regarding the representation of Laplace stretch. We employ a Gram–Schmidt procedure on this re-indexed deformation gradient $\boldsymbol{\mathcal{F}}$ and decompose it as

$$\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{R}}\,\boldsymbol{\mathcal{U}} \quad \text{where} \quad [\mathcal{U}_{ij}] = \begin{bmatrix} a & a\gamma & a\beta \\ 0 & b & b\alpha \\ 0 & 0 & c \end{bmatrix}. \tag{2}$$

The inverse of this rotation tensor, i.e., $\boldsymbol{\mathcal{R}}^T$, rotates the bases of an Eulerian frame of reference $\{\vec{\boldsymbol{e}}_i\}$ into the bases of our physical frame of reference denoted by $\{\vec{\boldsymbol{\mathcal{E}}}_i\}$. Each element of Laplace stretch has a direct physical meaning in this set of base vectors: a, b and c are elongations, while α, β and γ are shears. The upper-triangular Laplace stretch is related to the right Cauchy–Green tensor $\mathbf{C}$ through the relation $\mathbf{C} = \boldsymbol{\mathcal{U}}^T\boldsymbol{\mathcal{U}}$ and can be uniquely obtained from a given right Cauchy–Green tensor through its Cholesky factorization. Srinivasa (2012) further decomposed the Laplace stretch into two shearing motions, followed by an extensional motion along the directions $\{\vec{\boldsymbol{\mathcal{E}}}_i\}, i = 1,2,3$, thereby resulting in an Iwasawa (1949) decomposition of Laplace stretch. This decomposition is given by

$$[\mathcal{U}_{ij}] = \underbrace{\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}}_{\boldsymbol{\Lambda}} \underbrace{\begin{bmatrix} 1 & 0 & \beta \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{bmatrix}}_{\boldsymbol{\mathcal{U}}^{\alpha\beta}} \underbrace{\begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\boldsymbol{\mathcal{U}}^{\gamma}}. \tag{3}$$

The diagonal extensional deformation $\boldsymbol{\Lambda}$ can be further decomposed into a dilatation and three squeezes Freed (2017), viz.,

$$[\Lambda_{ij}] = \underbrace{\sqrt[3]{abc}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{dilatation}} \times \underbrace{\begin{bmatrix} \sqrt[3]{a/b} & 0 & 0 \\ 0 & \sqrt[3]{b/a} & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{1-2 planar sqeeze}} \times \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt[3]{b/c} & 0 \\ 0 & 0 & \sqrt[3]{c/b} \end{bmatrix}}_{\text{2-3 planar sqeeze}} \times \underbrace{\begin{bmatrix} \sqrt[3]{a/c} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt[3]{c/a} \end{bmatrix}}_{\text{3-1 planar sqeeze}}. \tag{4}$$

From this perspective, Laplace stretch is decomposed into seven modes of deformation: one dilatation, three squeezes, and three shears. These modes of deformation are useful when constructing our scalar, conjugate, stress/strain, base pairs. Note that the three squeezes are not independent; any one of the three, squeeze, stress/strain pairings can be expressed in terms of the other two.

Using the property that the set of all invertible, upper-triangular matrices forms a group under multiplication, Freed et al. (2019) decomposed the Laplace stretch into its elastic ($\boldsymbol{\mathcal{U}}^e$) and plastic ($\boldsymbol{\mathcal{U}}^p$) components. The matrices of $\boldsymbol{\mathcal{U}}^e$ and $\boldsymbol{\mathcal{U}}^p$ are both upper-triangular and their elements are given as

$$\boldsymbol{\mathcal{U}} = \boldsymbol{\mathcal{U}}^e\,\boldsymbol{\mathcal{U}}^p, \quad [\mathcal{U}^e_{ij}] = \begin{bmatrix} a^e & a^e\gamma^e & a^e\beta^e \\ 0 & b^e & b^e\alpha^e \\ 0 & 0 & c^e \end{bmatrix}, \quad [\mathcal{U}^p_{ij}] = \begin{bmatrix} a^p & a^p\gamma^p & a^p\beta^p \\ 0 & b^p & b^p\alpha^p \\ 0 & 0 & c^p \end{bmatrix}. \tag{5}$$

For example, the elastic components of $\boldsymbol{\mathcal{U}}^e$ can be obtained from $\boldsymbol{\mathcal{U}}$ and $\boldsymbol{\mathcal{U}}^p$ through the relations

$$\begin{aligned} a^e &= a/a^p, & \alpha^e &= b^p(\alpha - \alpha^p)/c^p, \\ b^e &= b/b^p, & \beta^e &= a^p\left[(\beta - \beta^p) - \alpha^p(\gamma - \gamma^p)\right]/c^p, \\ c^e &= c/c^p, & \gamma^e &= a^p(\gamma - \gamma^p)/b^p. \end{aligned} \tag{6}$$

Needless to say, one can derive the components of $\boldsymbol{\mathcal{U}}^p$ in terms of the elements of the total Laplace stretch and its elastic components in a similar fashion. Thus, among the pairs $(\boldsymbol{\mathcal{U}}, \boldsymbol{\mathcal{U}}^p)$, $(\boldsymbol{\mathcal{U}}^e, \boldsymbol{\mathcal{U}}^p)$, $(\boldsymbol{\mathcal{U}}, \boldsymbol{\mathcal{U}}^e)$, any one of them can serve as a set of primary kinematic variables. In this paper, we choose the pair $(\boldsymbol{\mathcal{U}}, \boldsymbol{\mathcal{U}}^p)$ as our primary kinematic variables due to their physical significance, which will be discussed shortly.

It is worth noting that because the matrix of $\boldsymbol{\mathcal{U}}^p$ is also upper-triangular, a decomposition similar to Eqs. (3) and (4) can be performed on $\boldsymbol{\mathcal{U}}^p$ with a superscript 'p' denoting the plastic part. Therefore, the matrix of $\boldsymbol{\mathcal{U}}^p$ can be decomposed as

$$[\mathcal{U}^p_{ij}] = \underbrace{\begin{bmatrix} a^p & 0 & 0 \\ 0 & b^p & 0 \\ 0 & 0 & c^p \end{bmatrix}}_{\boldsymbol{\Lambda}^p} \underbrace{\begin{bmatrix} 1 & 0 & \beta^p \\ 0 & 1 & \alpha^p \\ 0 & 0 & 1 \end{bmatrix}}_{\boldsymbol{\mathcal{U}}^{\alpha^p\beta^p}} \underbrace{\begin{bmatrix} 1 & \gamma^p & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\boldsymbol{\mathcal{U}}^{\gamma^p}} \tag{7}$$

with

$$[\Lambda^p_{ij}] = \underbrace{\sqrt[3]{a^pb^pc^p}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{dilatation}} \times \underbrace{\begin{bmatrix} \sqrt[3]{a^p/b^p} & 0 & 0 \\ 0 & \sqrt[3]{b^p/a^p} & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{1-2 planar sqeeze}} \times \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt[3]{b^p/c^p} & 0 \\ 0 & 0 & \sqrt[3]{c^p/b^p} \end{bmatrix}}_{\text{2-3 planar sqeeze}} \times \underbrace{\begin{bmatrix} \sqrt[3]{a^p/c^p} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt[3]{c^p/a^p} \end{bmatrix}}_{\text{3-1 planar sqeeze}}. \tag{8}$$

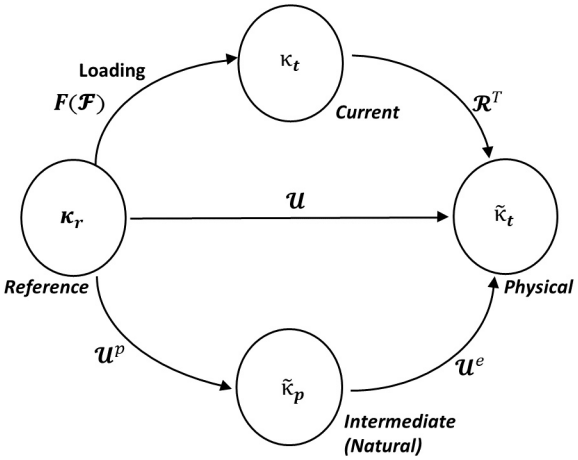

Fig. 1. Different configurations of the body and their associated maps.

This decomposition plays a key role in the construction of our constitutive model. It is worth noting that in metal plasticity, it is often assumed that the plastic deformation is a volume-preserving process, i.e., $\det(\mathcal{U}^p) = 1$. This assumption has a deeper consequence, because it enters into the constitutive equations separately as a constraint.

2.1. Natural configurations

An elastic–plastic decomposition of Laplace stretch assumes the existence of another intermediate, relaxed configuration $\tilde{\kappa}_p$, as shown in Fig. 1. The map $\mathcal{U}^p$ takes an infinitesimal fiber from a reference configuration of the body $\kappa_r(\mathcal{B})$ into the intermediate configuration $\tilde{\kappa}_p(\mathcal{B})$; whereas, the map $\mathcal{U}^e$ takes an infinitesimal fiber from $\tilde{\kappa}_p(\mathcal{B})$ and places it into its current configuration $\kappa_t(\mathcal{B})$. It is worth noting that a body occupies its configuration $\tilde{\kappa}_p(\mathcal{B})$ upon an elastic unloading from its current configuration, when viewed from our physical frame of reference $\tilde{\kappa}_t$, and thus, can be treated as one of the natural configurations, per the notation of Rajagopal and Srinivasa (1998a). Although it is not possible to elastically unload all of the infinitesimal parts of a body and fit them into an Euclidean space, this indeed becomes possible whenever this space is considered to be non-Euclidean. Therefore, the configuration $\tilde{\kappa}_p$ is essentially non-Euclidean. Paul and Freed (2020a) measured the torsion of this non-Euclidean space $\tilde{\kappa}_p(\mathcal{B})$ to characterize the geometrically necessary dislocations present within the body.

Thus, it is clearly understood that, in our physical frame of reference, the map $\mathcal{U}^p$ represents microstructural changes within the body; whereas, $\mathcal{U}^e$ represents an elastic deformation measured from the current natural configuration $\tilde{\kappa}_p$. In order to characterize a constitutive relation for an elastic–plastic material, we therefore have to determine two quantities: (*i*) a family of elastic responses measured from the current natural configuration $\tilde{\kappa}_p$, and (*ii*) an evolution equation for this natural configuration $\tilde{\kappa}_p$. This is achieved by specifying the constitutive assumptions for a Helmholtz potential ψ, and a rate of dissipation function ξ, and then employing the principle of maximum rate of dissipation. This rate of dissipation is required to be non-negative in order to identically satisfy the Clausius–Duhem inequality.

3. Constitutive modeling

In order to specify an elastic response measured from a current natural configuration and its evolution equation, we need to consider three thermodynamical quantities: (*i*) the stored energy characterized by a Helmholtz potential, (*ii*) the work done on an internal mass element, and (*iii*) a rate of dissipation function that measures the amount of mechanical work being converted into heat. The balance of energy equation stipulates that the rate of dissipation is obtained as the difference between a rate of change in the external work done and a rate of change in the Helmholtz potential. The rate of work done (or power) can easily be computed from the corresponding stress and strain attributes; whereas, one must specify a constitutive assumption for the Helmholtz potential function.

3.1. Stress power

Let us first define the velocity gradient associated with a Laplace stretch as $\mathcal{L} := \dot{\mathcal{U}}\,\mathcal{U}^{-1}$. Let $\mathbf{S}$ and $\mathbf{E}$ denote the symmetric, second, Piola–Kirchhoff stress and the Green strain, respectively. The rate of work done on an internal mass element is therefore given by Freed (2017)

$$\dot{W} := \mathrm{tr}(\mathbf{S}\,\dot{\mathbf{E}}) = \mathrm{tr}(\mathcal{S}\,\mathcal{L}) \tag{9}$$

where $\mathcal{S}$ is the Kirchhoff stress in our physical frame of reference $\tilde{\kappa}_t$, which is related to Eulerian Kirchhoff stress $\boldsymbol{\tau} := \det(\mathbf{F})\,\boldsymbol{\sigma}$ through the relation $\mathcal{S} := \mathcal{U}\,\boldsymbol{\tau}\,\mathcal{U}^T$ where $\boldsymbol{\sigma}$ denotes Cauchy stress. The stress tensor $\mathcal{S}$ is symmetric because Kirchhoff stress is symmetric.

Using the kinematic quantities described in Eqs. (3) and (4), the stress power can be expressed in terms of seven, conjugate, stress/strain, base pairs. Each of these base pairs describes a specific deformation mode. It is possible to establish bijective maps between these base pairs and the components of Kirchhoff stress $\boldsymbol{S}$, and the components of a velocity gradient expressed in terms of Laplace stretch $\boldsymbol{\mathcal{L}}$ (Freed, 2017). These maps are not necessarily unique. In this paper, we assume that the material exhibits an isotropic, elastic behavior. However, an anisotropic material behavior can easily be incorporated into the model through modification of these bijective maps. For the sake of completeness, the stress/strain base pairs corresponding to their respective modes of deformation are given below.

Pressure and dilatation:

$$\begin{aligned} &\text{Dilatation,} && \delta := \tfrac{1}{3}\ln(abc), \quad \dot{\delta} = \frac{1}{3}\left(\frac{\dot{a}}{a}+\frac{\dot{b}}{b}+\frac{\dot{c}}{c}\right) \\ &\text{Pressure,} && \pi := S_{11}+S_{22}+S_{33} \end{aligned} \tag{10a}$$

Squeeze:

1-2 planar squeeze

$$\begin{aligned} &\text{Strain,} && \varepsilon_1 := \tfrac{1}{3}\ln(a/b), \quad \dot{\varepsilon_1} = \frac{1}{3}\left(\frac{\dot{a}}{a}-\frac{\dot{b}}{b}\right) \\ &\text{Conjugate stress,} && \sigma_1 := S_{11}-S_{22} \end{aligned} \tag{10b}$$

2-3 planar squeeze

$$\begin{aligned} &\text{Strain,} && \varepsilon_2 := \tfrac{1}{3}\ln(b/c), \quad \dot{\varepsilon_2} = \frac{1}{3}\left(\frac{\dot{b}}{b}-\frac{\dot{c}}{c}\right) \\ &\text{Conjugate stress,} && \sigma_2 := S_{22}-S_{33} \end{aligned} \tag{10c}$$

3-1 planar squeeze

$$\begin{aligned} &\text{Strain,} && \varepsilon_3 := \tfrac{1}{3}\ln(c/a), \quad \dot{\varepsilon_3} = \frac{1}{3}\left(\frac{\dot{c}}{c}-\frac{\dot{a}}{a}\right) \\ &\text{Conjugate stress,} && \sigma_3 := S_{33}-S_{11} \end{aligned} \tag{10d}$$

Note that the stresses $\sigma_1, \sigma_2, \sigma_3$, their corresponding conjugate strains, and their rates are not independent; specifically, any one of these stress/strain pairs for squeeze can be expressed as a linear combination of the other two, e.g., $\sigma_3 = -(\sigma_1+\sigma_2)$ and $\varepsilon_3 = -(\varepsilon_1+\varepsilon_2)$. The stress/strain base pairs given in Eqs. (10a)–(10d) come from the extensional part of the Laplace stretch, $\boldsymbol{\Lambda}$, whereas the remaining three stress/strain base pairs correspond to the three shear deformations, and are given as

Shear:

Out-of plane shears:

$$\text{Strain, } \gamma_1 := \alpha \quad \text{Conjugate stress, } \tau_1 := \frac{b}{c}S_{23}; \tag{10e}$$

$$\text{Strain, } \gamma_2 := \beta \quad \text{Conjugate stress, } \tau_2 := \frac{a}{c}S_{13}; \tag{10f}$$

In-plane shear:

$$\text{Strain, } \gamma_3 := \gamma \quad \text{Conjugate stress, } \tau_3 := \frac{a}{b}S_{12} - \alpha\frac{a}{c}S_{13}. \tag{10g}$$

Note that a coupling exists between the in-plane and an out-of-plane shear. Now, using the stress/strain base pairs given in Eqs. (10a)–(10g), the stress power can be rewritten as

$$\dot{W} = \pi\dot{\delta} + \sum_{i=1}^{3}\sigma_i\dot{\varepsilon}_i + \sum_{i=1}^{3}\tau_i\dot{\gamma}_i. \tag{11}$$

Therefore, instead of the traditionally used tensor invariants, here we can use the list of scalar variables $l_{\mathcal{V}}$ defined as

$$l_{\mathcal{V}} := \{\delta \quad \varepsilon_1 \quad \varepsilon_2 \quad \varepsilon_3 \quad \gamma_1 \quad \gamma_2 \quad \gamma_3\} \tag{12}$$

as our primary kinematic variables. In a similar way, we can also define a list of kinematic variables containing the rate of strain attributes as

$$l_{\dot{\mathcal{V}}} := \{\dot{\delta} \quad \dot{\varepsilon}_1 \quad \dot{\varepsilon}_2 \quad \dot{\varepsilon}_3 \quad \dot{\gamma}_1 \quad \dot{\gamma}_2 \quad \dot{\gamma}_3\} \ . \tag{13}$$

Because, e.g., strain measure ε_3 and its rate $\dot{\varepsilon}_3$ can be expressed as a linear combination of the other two squeeze strains and strain-rates, one may have a natural propensity to exclude them from lists $l_{\mathcal{V}}$ and $l_{\dot{\mathcal{V}}}$, respectively. However, if this were to be done, then it would become particularly difficult to track the appropriate strains and strain-rates arising within a constitutive relation given a particular function, e.g., the Helmholtz potential or the dissipation function. Keeping this in mind, we will include all three squeeze strains ε_i and their rates $\dot{\varepsilon}_i$ in our lists of kinematic variables. The stress attributes conjugate to the strain attributes $l_{\mathcal{V}}$, can be listed as

$$l_\sigma := \{\ \pi \quad \sigma_1 \quad \sigma_2 \quad \sigma_3 \quad \tau_1 \quad \tau_2 \quad \tau_3\}\ . \tag{14}$$

Hill (1978) also used six stress/strain base pairs to construct a constitutive model for inelastic materials. In his work, the kinematic attributes are defined based on the components of a strain measure of (second-order) tensorial nature, with the kinetic attributes being work-conjugates to these kinematic (or, strain) attributes; they are Voigt components. On the other hand, in the present work, the strain attributes come from a decomposition of the upper-triangular Laplace stretch, and each of these strain attributes corresponds to a specific deformation mode. As with Hill's formulation, our stress attributes are work-conjugates to our strain attributes.

Two similar sets of kinematic variables and their rates can be defined for the elastic and plastic components of Laplace stretch. These lists of variables are given as

$$l_{\mathcal{U}^p} := \{\ \delta^p \quad \varepsilon_1^p \quad \varepsilon_2^p \quad \varepsilon_3^p \quad \gamma_1^p \quad \gamma_2^p \quad \gamma_3^p\} \tag{15}$$

$$l_{\mathcal{U}^e} := \{\delta^e \quad \varepsilon_1^e \quad \varepsilon_2^e \quad \varepsilon_3^e \quad \gamma_1^e \quad \gamma_2^e \quad \gamma_3^e\} \tag{16}$$

and

$$l_{\dot{\mathcal{U}}^p} := \{\dot{\delta}^p \quad \dot{\varepsilon}_1^p \quad \dot{\varepsilon}_2^p \quad \dot{\varepsilon}_3^p \quad \dot{\gamma}_1^p \quad \dot{\gamma}_2^p \quad \dot{\gamma}_3^p\} \tag{17}$$

$$l_{\dot{\mathcal{U}}^e} := \{\dot{\delta}^e \quad \dot{\varepsilon}_1^e \quad \dot{\varepsilon}_2^e \quad \dot{\varepsilon}_3^e \quad \dot{\gamma}_1^e \quad \dot{\gamma}_2^e \quad \dot{\gamma}_3^e\} \tag{18}$$

where the elastic strain attributes are defined as

$$\begin{gathered}\delta^e = \tfrac{1}{3}\ln(a^e b^e c^e), \quad \varepsilon_1^e = \tfrac{1}{3}\ln(a^e/b^e), \quad \varepsilon_2^e = \tfrac{1}{3}\ln(b^e/c^e), \quad \varepsilon_3^e = \tfrac{1}{3}\ln(c^e/a^e),\\ \gamma_1^e = \alpha^e, \quad \gamma_2^e = \beta^e, \quad \gamma_3^e = \gamma^e\end{gathered} \tag{19}$$

and their plastic counterparts are given as

$$\begin{gathered}\delta^p = \tfrac{1}{3}\ln(a^p b^p c^p), \quad \varepsilon_1^p = \tfrac{1}{3}\ln(a^p/b^p), \quad \varepsilon_2^p = \tfrac{1}{3}\ln(b^p/c^p), \quad \varepsilon_3^p = \tfrac{1}{3}\ln(c^p/a^p),\\ \gamma_1^p = \alpha^p, \quad \gamma_2^p = \beta^p, \quad \gamma_3^p = \gamma^p.\end{gathered} \tag{20}$$

Using relationships between components of the total Laplace stretch and those of their elastic and plastic counterparts, given in Eq. (6), one can establish a relationship between the total strain attributes and their elastic and plastic components; specifically,

$$\begin{aligned}\delta &= \delta^e + \delta^p,\\ \varepsilon_1 &= \varepsilon_1^e + \varepsilon_1^p,\\ \varepsilon_2 &= \varepsilon_2^e + \varepsilon_2^p,\\ \varepsilon_3 &= \varepsilon_3^e + \varepsilon_3^p,\\ \gamma_1 &= \exp(-3\varepsilon_1^p)\,\gamma_1^e + \gamma_1^p,\\ \gamma_2 &= \exp(3\varepsilon_3^p)\,\gamma_2^e + \gamma_2^p + \gamma_1^p\,(\gamma_3 - \gamma_3^p),\\ \gamma_3 &= \exp(-3\varepsilon_1^p)\,\gamma_3^e + \gamma_3^p.\end{aligned} \tag{21}$$

Traditionally, an additive decomposition of the total strain into its elastic and plastic components is more commonly used in a small displacement-gradient theory; whereas, a multiplicative decomposition of a kinematic quantity, such as a deformation gradient, is typically used in a finite deformation setting.[1] Here, interestingly, an additive decomposition of the total strain results as a direct consequence of the multiplicative elastic–plastic decomposition of the Laplace stretch, without any assumption of a small displacement gradient. This key feature of **QR** kinematics is extremely useful when constructing constitutive models for elastic–plastic materials due to its similarity with small strain theory. Such an additive elastic–plastic decomposition of total strain was also achieved by Miehe (1998) for a finite deformation theory through the assumption of a plastic metric. Note that although the additive decompositions of the dilatational strain δ and the squeeze strains ε_i are rather straightforward, such is not the case for the shear strains γ_i. Specifically, the elastic components of the shear strains appear as a product with a function of the squeeze strains. Although this fact does not pose much of a problem in our subsequent derivations, one must be vigilant whenever one is dealing with the shear terms.

If $\boldsymbol{\mathcal{L}}^p$ denotes a plastic velocity gradient, defined as $\boldsymbol{\mathcal{L}}^p := \dot{\boldsymbol{\mathcal{U}}}^p\,\boldsymbol{\mathcal{U}}^{p-1}$, then it is possible to establish a bijective map between the elements of the matrix of $\boldsymbol{\mathcal{L}}^p$ and the list of variables $l_{\dot{\mathcal{U}}^p}$. Such a map is not, in general, unique. This map can be written as

$$\begin{pmatrix}\dot{\delta}^p\\ \dot{\varepsilon}_1^p\\ \dot{\varepsilon}_2^p\\ \dot{\gamma}_1^p\\ \dot{\gamma}_2^p\\ \dot{\gamma}_3^p\end{pmatrix} = \begin{bmatrix}1/3 & 1/3 & 1/3 & 0 & 0 & 0\\ 1/3 & -1/3 & 0 & 0 & 0 & 0\\ 0 & 1/3 & -1/3 & 0 & 0 & 0\\ 0 & 0 & 0 & c^p/b^p & 0 & 0\\ 0 & 0 & 0 & 0 & c^p/a^p & b^p a^p/a^p\\ 0 & 0 & 0 & 0 & 0 & b^p/a^p\end{bmatrix}\begin{pmatrix}\mathcal{L}_{11}^p\\ \mathcal{L}_{22}^p\\ \mathcal{L}_{33}^p\\ \mathcal{L}_{23}^p\\ \mathcal{L}_{13}^p\\ \mathcal{L}_{12}^p\end{pmatrix}. \tag{22}$$

[1] Although an additive elastic–plastic decomposition of the rate of deformation tensor is quite commonly used in finite deformation theory, e.g., cf. Nemat-Nasser (1982).

Therefore, one can potentially replace $\boldsymbol{\mathcal{L}}^p$ by the list of plastic conjugate strain rates $l_{\dot{\mathcal{U}}^p}$.

3.2. Elastic domain and the Helmholtz potential function

We assume that for each natural configuration $\tilde{\kappa}_p$ there also exists a non-empty elastic domain. If the Green strain $\mathbf{E} = 1/2(\mathbf{C} - \mathbf{I})$ lies within the elastic domain, the plastic velocity gradient must be zero, because $\boldsymbol{\mathcal{U}}^p$ essentially represents a microstructural change, i.e., evolution of the natural configuration $\tilde{\kappa}_p$. Therefore, for a fixed natural configuration $\tilde{\kappa}_p$, the elastic domain can be represented by

$$\boldsymbol{\mathcal{L}}^p = \mathbf{0}. \tag{23}$$

In view of the bijective map in Eq. (22), the elastic domain for a fixed natural configuration can also be characterized as

$$l_{\dot{\mathcal{U}}^p} = \mathbf{0} \implies \dot{\delta}^p = \dot{\varepsilon}_1^p = \dot{\varepsilon}_2^p = \dot{\varepsilon}_3 = \dot{\gamma}_1^p = \dot{\gamma}_2^p = \dot{\gamma}_3^p = 0. \tag{24}$$

Note that the tensor equation (23) reduces to a set of six scalar equations when represented in terms of these plastic strain rates.

Now, for each fixed natural configuration $\tilde{\kappa}_p$, we assume that the elastic response is characterized by a Helmholtz potential function ψ, which depends upon the deformation of a body measured from its reference configuration κ_r and its natural configuration $\tilde{\kappa}_p$, i.e., the Laplace stretch $\boldsymbol{\mathcal{U}}$ and its plastic component $\boldsymbol{\mathcal{U}}^p$. Therefore, the Helmholtz potential has the form

$$\psi = \overline{\psi}\,(\boldsymbol{\mathcal{U}}, \boldsymbol{\mathcal{U}}^p). \tag{25}$$

In view of Section 3.1, the tensor arguments of ψ in Eq. (25) can be replaced by the lists of scalar strain bases $l_{\mathcal{U}}$ and $l_{\mathcal{U}^p}$. Thus, the Helmholtz potential takes on a functional form of

$$\psi = \hat{\psi}\,(l_{\mathcal{U}}, l_{\mathcal{U}^p}) = \hat{\psi}\,(\delta, \varepsilon_1, \varepsilon_2, \varepsilon_3, \gamma_1, \gamma_2, \gamma_3, \delta^p, \varepsilon_1^p, \varepsilon_2^p, \varepsilon_3^p, \gamma_1^p, \gamma_2^p, \gamma_3^p). \tag{26}$$

3.3. The rate of dissipation function

The isothermal energy balance equation stipulates that the rate of dissipation, ξ, is equal to a difference between the mechanical power and a rate of change in the Helmholtz potential function. Therefore, the rate of dissipation can be written as

$$\xi := \dot{W} - \rho_0 \dot{\psi} \geq 0. \tag{27}$$

The non-negativity of ξ is to ensure that some form of the rate of dissipation inequality (e.g., Clausius–Duhem inequality) is identically satisfied. Because the dissipation of mechanical energy into heat is associated with changes in microstructure, (i.e., evolution of the natural configuration $\tilde{\kappa}_p$) and therefore, the plastic deformation of a body, it is reasonable to assume that the rate of dissipation function is dependent upon the plastic components of Laplace stretch and their rates, or in other words, on the strain measures $l_{\mathcal{U}^p}$ and their rates $l_{\dot{\mathcal{U}}^p}$, as described in Section 3.2. For now, we further assume that the dissipation function is a closed, bounded and continuously differentiable function. The differentiability of the rate of dissipation function has an important significance, which will be discussed shortly. Therefore, the rate of dissipation function ξ is considered to have functional form of

$$\xi = \hat{\xi}(l_{\mathcal{U}^p}, l_{\dot{\mathcal{U}}^p}) = \hat{\xi}\,(\delta^p, \varepsilon_1^p, \varepsilon_2^p, \varepsilon_3^p, \gamma_1^p, \gamma_2^p, \gamma_3^p, \dot{\delta}^p, \dot{\varepsilon}_1^p, \dot{\varepsilon}_2^p, \dot{\varepsilon}_3^p, \dot{\gamma}_1^p, \dot{\gamma}_2^p, \dot{\gamma}_3^p). \tag{28}$$

For the sake of simplicity, here we have assumed that the material is perfectly plastic, i.e., it does not exhibit any hardening or softening behavior. Such material behavior can easily be incorporated into the model by considering additional variables in the arguments of the rate of dissipation, ξ, and by specifying their evolution equations. The elastic response of a body for a fixed natural configuration $\tilde{\kappa}_p$ is completely non-dissipative. Therefore, in view of Section 3.2, we conclude that

$$\hat{\xi}\,(l_{\mathcal{U}^p}, \{0,0,0,0,0,0,0\}) = 0. \tag{29}$$

Now, substituting Eqs. (11) and (26) into Eq. (28), we obtain

$$\begin{aligned}
&\left(\pi - \rho_0 \frac{\partial \hat{\psi}}{\partial \delta}\right)\dot{\delta} + \left(\sigma_1 - \rho_0 \frac{\partial \hat{\psi}}{\partial \varepsilon_1}\right)\dot{\varepsilon}_1 + \left(\sigma_2 - \rho_0 \frac{\partial \hat{\psi}}{\partial \varepsilon_2}\right)\dot{\varepsilon}_2 + \left(\sigma_3 - \rho_0 \frac{\partial \hat{\psi}}{\partial \varepsilon_3}\right)\dot{\varepsilon}_3 \\
&+ \left(\tau_1 - \rho_0 \frac{\partial \hat{\psi}}{\partial \gamma_1}\right)\dot{\gamma}_1 + \left(\tau_2 - \rho_0 \frac{\partial \hat{\psi}}{\partial \gamma_2}\right)\dot{\gamma}_2 + \left(\tau_3 - \rho_0 \frac{\partial \hat{\psi}}{\partial \gamma_3}\right)\dot{\gamma}_3 = \hat{\xi} \\
&+ \rho_0 \left(\frac{\partial \hat{\psi}}{\partial \delta^p}\dot{\delta}^p + \frac{\partial \hat{\psi}}{\partial \varepsilon_1^p}\dot{\varepsilon}_1^p + \frac{\partial \hat{\psi}}{\partial \varepsilon_2^p}\dot{\varepsilon}_2^p + \frac{\partial \hat{\psi}}{\partial \varepsilon_3^p}\dot{\varepsilon}_3^p + \rho_0 \frac{\partial \hat{\psi}}{\partial \gamma_1}\dot{\gamma}_1 + \rho_0 \frac{\partial \hat{\psi}}{\partial \gamma_2}\dot{\gamma}_2 + \rho_0 \frac{\partial \hat{\psi}}{\partial \gamma_3}\dot{\gamma}_3\right)
\end{aligned} \tag{30}$$

where the hats on top of ψ and ξ imply that they are expressed in terms of their respective conjugate strain measures according to Eqs. (26) and (28).

We notice that the left-hand side of Eq. (30) is a function of the list of variables $l_{\mathcal{U}}$ and $l_{\dot{\mathcal{U}}}$ for a given natural configuration $\tilde{\kappa}_p$, whereas the right-hand side is independent of those variables. If the elastic response of the material is assumed to be that of a Green elastic solid, then the stress attributes can be written as

$$\pi = \rho_0 \frac{\partial \hat{\psi}}{\partial \delta}, \qquad \sigma_i = \rho_0 \frac{\partial \hat{\psi}}{\partial \varepsilon_i}, \qquad \tau_i = \rho_0 \frac{\partial \hat{\psi}}{\partial \gamma_i} \tag{31}$$

where $i = 1, 2, 3$. Using the interdependence of the squeeze stress/strain pairs (specifically, $\sigma_3 = -(\sigma_1 + \sigma_2)$ and $\varepsilon_3 = -(\varepsilon_1 + \varepsilon_2)$), Eq. (31) can be alternatively written as

$$\pi = \rho_0 \frac{\partial \hat{\psi}}{\partial \delta}, \tag{32a}$$

$$2\sigma_1 + \sigma_2 = \rho_0 \frac{\partial \hat{\psi}}{\partial \varepsilon_1}, \tag{32b}$$

$$\sigma_1 + 2\sigma_2 = \rho_0 \frac{\partial \hat{\psi}}{\partial \varepsilon_2}, \tag{32c}$$

$$\tau_i = \rho_0 \frac{\partial \hat{\psi}}{\partial \gamma_i}, \quad i = 1, 2, 3. \tag{32d}$$

The assumption of an elastic response stipulates that the rate of dissipation function satisfies the constraint

$$\hat{\xi} = -\rho_0 \left[\frac{\partial \hat{\psi}}{\partial \delta^p} \dot{\delta}^p + \sum_{i=1}^{3} \left(\frac{\partial \hat{\psi}}{\partial \varepsilon_i^p} \dot{\varepsilon}_i^p + \frac{\partial \hat{\psi}}{\partial \gamma_i} \dot{\gamma}_i \right) \right]. \tag{33}$$

Eqs. (32a)–(32d) imply that the stresses are derivable from the Helmholtz potential ψ. A similar result was obtained by Freed (2017) for the elastic case. However, the main deviation from Freed's result is that in the elastoplastic case the Helmholtz potential is also a function of the variables listed in $l_{\mathcal{U}^p}$, and thus, is dependent upon the natural configuration $\tilde{\kappa}_p$ and its evolution.

3.4. Maximization of the rate of dissipation

It is well-known that although a satisfaction of the second law of thermodynamics is a necessary condition for any valid constitutive model, it is not sufficient for determining an evolution equation for the natural configurations. In our theory, the second law of thermodynamics is identically satisfied though an assumption of the non-negativity of ξ. However, we need to make a more stringent assumption to determine the list of variables $l_{\mathcal{U}^p}$, which in turn serves as an evolution equation for $\tilde{\kappa}_p$. Here we adopt the criterion for a maximum rate of dissipation that states that *out of all admissible values in the list of variables $l_{\mathcal{U}^p}$ (alternatively, the plastic velocity gradient $\mathcal{L}^p$), the one that maximizes the rate of dissipation, while satisfying the reduced rate of dissipation equation (33), is the one that governs evolution of the natural configuration $\tilde{\kappa}_p$*. This criterion was proposed by Rajagopal and Srinivasa (1998b, 2004) and can be viewed as an extension of Onsager's minimum rate of entropy production criterion and Ziegler's normality rule. Therefore, mathematically, the determination of $l_{\mathcal{U}^p}$ becomes a constrained optimization problem with respect to the list of variables $l_{\mathcal{U}^p}$, with the rate of dissipation ξ as its objective function and the reduced rate of dissipation equation (33) as a constraint. If ξ is assumed to be a sufficiently smooth function in the plastic strain-rate domain, then the problem can be carried out by using a traditional Lagrange multiplier approach. The solution yields a set of seven scalar equations, each corresponding to a mode of deformation, given by

$$\frac{\partial \hat{\xi}}{\partial \dot{\delta}^p} = -\lambda\, \rho_0 \frac{\partial \hat{\psi}}{\partial \delta^p}; \quad \frac{\partial \hat{\xi}}{\partial \dot{\varepsilon}_i^p} = -\lambda\, \rho_0 \frac{\partial \hat{\psi}}{\partial \varepsilon_i^p}; \quad \frac{\partial \hat{\xi}}{\partial \dot{\gamma}_i^p} = -\lambda\, \rho_0 \frac{\partial \hat{\psi}}{\partial \gamma_i^p} \tag{34}$$

where $i = 1, 2, 3$, with $\lambda = -\overline{\lambda}/(1+\overline{\lambda})$ wherein $\overline{\lambda}$ is a Lagrange multiplier. It is possible to evaluate λ through a satisfaction of constraint equation (33).

3.5. Special form for ψ and rate independent plasticity

Notice that Eq. (34) is a set of seven implicit equations for the plastic strain rates listed in $l_{\mathcal{U}^p}$. Such implicit equations are difficult to solve and do not provide much physical interpretations. Therefore, it is instructive to choose a specific form for the Helmholtz potential function at this juncture. As mentioned earlier, the Helmholtz potential ψ characterizes the elastic response of a material for a fixed natural configuration, and hence, it is common practice to consider the Helmholtz potential as a function of the elastic strain attributes (e.g., Srinivasa, 2010), i.e.,

$$\psi = \hat{\psi}(\delta^e, \varepsilon_1^e, \varepsilon_2^e, \varepsilon_3^e, \gamma_1^e, \gamma_2^e, \gamma_3^e). \tag{35}$$

However, in view of Eq. (21), one can write the elastic strain attributes in terms of the total strain attributes and their plastic components. In other words, the difference between the total strain attributes and their corresponding plastic components always represent the elastic strain attributes, or linear combinations thereof. Therefore, because the Helmholtz potential function is to be specified for a fixed natural configuration, it is reasonable to assume that is a homogeneous, quadratic function of the difference between the total strain attributes and their corresponding plastic components. Specifically, one can write

$$\begin{aligned} \psi = \hat{\psi}(l_{\mathcal{U}^e}, l_{\mathcal{U}^p}) = \frac{1}{2} \Big[& N_{00} (\delta - \delta^p)^2 + \sum_{i=1}^{3} N_{0i} (\delta - \delta^p)(\varepsilon_i - \varepsilon_i^p) + \sum_{i=1}^{3} N_{0(i+3)} (\delta - \delta^p)(\gamma_i - \gamma_i^p) \\ & + \sum_{\substack{i,j=1 \\ i \le j}}^{3} N_{ij} (\varepsilon_i - \varepsilon_i^p)(\varepsilon_j - \varepsilon_j^p) + \sum_{i,j=1}^{3} N_{i(j+3)} (\varepsilon_i - \varepsilon_i^p)(\gamma_j - \gamma_j^p) \\ & + \sum_{\substack{i,j=1 \\ i \le j}}^{3} N_{(i+3)(j+3)} (\gamma_i - \gamma_i^p)(\gamma_j - \gamma_j^p) \Big]. \end{aligned} \tag{36}$$

where the N's are material parameters. These material parameters are not all independent. In fact, for an isotropic material, these material parameters can be expressed in terms of two, independent Lamé constants. The Helmholtz potential is chosen in this way to avoid any complication in analysis that may arise due to couplings between the normal and shear plastic-strain attributes whenever their elastic counterparts are expressed in terms of the sets of variables $l_{\mathcal{U}^e}$ and $l_{\mathcal{U}^p}$. Moreover, the form for ψ in Eq. (36) essentially leads to a Green elastic solid response[2] measured from a fixed natural configuration $\tilde{\kappa}_p$.

The choice of this special form has a deeper consequence when developing the rest of our constitutive model. From this assumed form for ψ, by simple calculations, one can easily obtain

$$\begin{aligned}\pi &= \rho_0 \frac{\partial \hat{\psi}}{\partial \delta} = -\rho_0 \frac{\partial \hat{\psi}}{\partial \delta^p},\\ \sigma_i &= \rho_0 \frac{\partial \hat{\psi}}{\partial \epsilon_i} = -\rho_0 \frac{\partial \hat{\psi}}{\partial \epsilon_i^p},\\ \tau_i &= \rho_0 \frac{\partial \hat{\psi}}{\partial \gamma_i} = -\rho_0 \frac{\partial \hat{\psi}}{\partial \gamma_i^p}.\end{aligned} \tag{37}$$

Now, substituting these relations into the reduced rate of dissipation equation (34), we obtain

$$\pi\, \dot{\delta}^p + \sum_{i=1}^{3} \left(\sigma_i\, \dot{\epsilon}_i^p + \tau_i\, \dot{\gamma}_i^p\right) = \xi. \tag{38}$$

This is the reduced rate of equation for our chosen form for the Helmholtz potential function, and should be used as a constraint for the subsequent development of our constitutive model. For future reference, we note that because Eq. (37) is invertible, the rate of dissipation function ξ can be expressed in terms of the stress attributes and the list of variables $l_{\dot{\mathcal{U}}^p}$, i.e., $\xi = \bar{\xi}(l_\sigma, l_{\dot{\mathcal{U}}^p})$. Therefore, it is now possible to carry out a maximization of the rate of dissipation function subject to the constraint (38) and taken with respect to either stress or plastic strain-rate attributes, thereby making an inversion of the relations (34) possible.

Yield condition

Before obtaining an evolution equation for the natural configurations $\tilde{\kappa}_p$, we need to determine under what conditions the natural configurations change (or the material yields). Note that our theory does not presuppose the existence of a yield surface, as is the case in classical plasticity theory. In fact, whether the material under consideration shows yielding behavior or a creep-like behavior is determined by the nature of the rate of dissipation function. As we mentioned earlier, whenever the variables listed in $l_{\dot{\mathcal{U}}^p}$ are all zero, the dissipation function vanishes and the material exhibits an elastic response. However, if the dissipation function is considered to be a positively homogeneous function of order 1, then it is non-differentiable whenever $l_{\dot{\mathcal{U}}^p}$ is zero. In fact, a dissipation function that is non-differentiable at $l_{\dot{\mathcal{U}}^p} = \mathbf{0}$ corresponds to a material with a definite yield surface, whereas a sufficiently smooth dissipation function (even at $l_{\dot{\mathcal{U}}^p} = \mathbf{0}$) leads to creep-like material behavior. Clearly, for materials showing yielding behavior, one cannot use a Lagrange multiplier technique to maximize the rate of dissipation function ξ. Nevertheless, it is possible to define a suitable yield function for both these cases.

It is to note that the maximum rate of dissipation criterion stipulates that whenever a dissipative process is possible, it will occur. In other words, a body undergoes a non-dissipative process *only when* there is no possibility for a dissipative process. In general, the elastic domain of a material can be characterized by those values of l_σ for which all the plastic strain-rate attributes are zero. Notice that a satisfaction of Eq. (38) is crucial to the occurrence of a dissipative process. Specifically, the only admissible nonzero values of $l_{\dot{\mathcal{U}}^p}$ are those that satisfy the reduced rate of dissipation equation (38). Among these admissible values, the ones that maximize the rate of dissipation function $\xi = \bar{\xi}(l_\sigma, l_{\dot{\mathcal{U}}^p})$ are chosen as the 'correct' plastic strain-rate attributes. Therefore, we can alternatively[3] characterize an elastic domain based upon whether Eq. (38) is satisfied. Clearly, from Eq. (38), we can conclude that $l_\sigma = \mathbf{0}$ belongs to this set of values. Because the dissipation function ξ is bounded, closed and continuous on the plastic strain-rate attributes $l_{\dot{\mathcal{U}}^p}$ for a prescribed set of values for l_σ, it is always possible to find values for the plastic strain-rate attributes such that

$$\pi\, \dot{\delta}^p + \sigma_1\, \dot{\epsilon}_1^p + \sigma_2\, \dot{\epsilon}_2^p + \sigma_3\, \dot{\epsilon}_3^p + \tau_1\, \dot{\gamma}_1^p + \tau_2\, \dot{\gamma}_2^p + \tau_3\, \dot{\gamma}_3^p < \bar{\xi}(l_\sigma, l_{\dot{\mathcal{U}}^p}). \tag{39}$$

For these values of $l_{\dot{\mathcal{U}}^p}$, because the reduced rate of dissipation equation (38) is violated, we can conclude that for these values of $l_{\dot{\mathcal{U}}^p}$, the material exhibits an elastic (i.e., non-dissipative) response. We can now formally introduce a yield function for the material. For a given l_σ, let us define a function $Y(l_{\dot{\mathcal{U}}^p})$ as

$$Y(l_{\dot{\mathcal{U}}^p}) := \max_{l_{\dot{\mathcal{U}}^p} \neq \mathbf{0}} \frac{\pi\, \dot{\delta}^p + \sigma_1\, \dot{\epsilon}_1^p + \sigma_2\, \dot{\epsilon}_2^p + \sigma_3\, \dot{\epsilon}_3^p + \tau_1\, \dot{\gamma}_1^p + \tau_2\, \dot{\gamma}_2^p + \tau_3\, \dot{\gamma}_3^p}{\bar{\xi}(l_\sigma, l_{\dot{\mathcal{U}}^p})}. \tag{40}$$

Whenever $Y(l_{\dot{\mathcal{U}}^p}) = 1$, it is possible to find a combination for the sets of values of l_σ and $l_{\dot{\mathcal{U}}^p}$ that satisfies the reduced rate of dissipation equation (38), and thus, this condition provides a set of admissible, nonzero values for the plastic strain-rate attributes $l_{\dot{\mathcal{U}}^p}$. Therefore, the yield condition is defined as $Y(l_{\dot{\mathcal{U}}^p}) = 1$ with function $Y(l_{\dot{\mathcal{U}}^p})$ being referred to as the yield function. Clearly, if $Y(l_{\dot{\mathcal{U}}^p}) < 1$, then equation (39) is satisfied, and as such, the material response is elastic. It has been shown that this yield function is convex in strain-rate space (see Appendix).

[2] I.e., the response of a hyperelastic solid.

[3] Note that here the stress attributes l_σ are held fixed and $l_{\dot{\mathcal{U}}^p}$ are allowed to vary.

Normality rule

We now derive a key result pertinent to our constitutive theory. If the rate of dissipation function ξ is assumed to be a continuously differentiable function, i.e., the material exhibits a creep-like behavior, then substitution of the relations (37) into Eq. (34) produces

$$\pi = \frac{1}{\lambda}\,\frac{\partial\hat{\xi}}{\partial\dot{\delta}^p}, \qquad \sigma_i = \frac{1}{\lambda}\,\frac{\partial\hat{\xi}}{\partial\dot{\varepsilon}_i^p}, \qquad \tau_i = \frac{1}{\lambda}\,\frac{\partial\hat{\xi}}{\partial\dot{\gamma}_i^p} \tag{41}$$

where $i = 1, 2, 3$. Therefore, from the above equations, one can conclude that *the stress attributes are directed along the gradient of the rate of dissipation function with respect to their corresponding plastic strain-rate attributes*. Clearly, Eq. (41) is equivalent to the normality rule used in classical plasticity theory whenever $l_{\dot{\mathcal{V}}^p} \neq \mathbf{0}$.

Note that the set of Eqs. (41) is valid only when $l_{\dot{\mathcal{V}}^p} \neq \mathbf{0}$, i.e., whenever the material is undergoing plastic deformation. In this case, the reduced rate of dissipation equation (38) must be satisfied. Now, the Lagrange multiplier λ can be evaluated by substituting the relations (41) into the reduced rate of dissipation equation (38) and is given as

$$\lambda = \frac{1}{\xi}\left[\dot{\delta}^p\,\frac{\partial\xi}{\partial\dot{\delta}^p} + \sum_{i=1}^{3}\left(\dot{\varepsilon}_i^p\,\frac{\partial\xi}{\partial\dot{\varepsilon}_i^p} + \dot{\gamma}_i^p\,\frac{\partial\xi}{\partial\dot{\gamma}_i^p}\right)\right]. \tag{42}$$

For the materials that exhibit yielding behavior, the rate of dissipation function is no longer differentiable with respect to the plastic strain-rate attributes at the yield surface. Therefore, we must adopt a standard method from convex analysis, instead of employing the Lagrange multiplier technique, to derive similar results for this case. Let us consider a list of prescribed stress attributes l_σ and two plastic strain-rate attributes: (*i*) $\bar{l}_{\dot{\mathcal{V}}^p}$ that satisfy the yield condition $Y(\bar{l}_{\dot{\mathcal{V}}^p}) = 1$ and (*ii*) $l_{\dot{\mathcal{V}}^p}$ for which $Y(l_{\dot{\mathcal{V}}^p}) < 1$. Now, since the rate of dissipation function is assumed to be a bounded, closed, homogeneous function of order 1 in the plastic strain-rate attribute space, and among the two processes only the former is dissipative, it can easily be concluded that for given stress attributes $\xi(l_\sigma, l_{\dot{\mathcal{V}}^p}) \leq \xi(l_\sigma, \bar{l}_{\dot{\mathcal{V}}^p})$.

Now, in view of the expression for the yield condition, it is easily understood that

$$\frac{\pi\,\dot{\delta}^p + \sum_{i=1}^{3}\left(\sigma_i\,\dot{\varepsilon}_i^p + \tau_i\,\dot{\gamma}_i^p\right)}{\xi(l_\sigma, l_{\dot{\mathcal{V}}^p})} \leq \frac{\pi\,\dot{\bar{\delta}}^p + \sum_{i=1}^{3}\left(\sigma_i\,\dot{\bar{\varepsilon}}_i^p + \tau_i\,\dot{\bar{\gamma}}_i^p\right)}{\xi(l_\sigma, \bar{l}_{\dot{\mathcal{V}}^p})} = 1. \tag{43}$$

Using the fact that $\xi(l_\sigma, l_{\dot{\mathcal{V}}^p}) \leq \xi(l_\sigma, \bar{l}_{\dot{\mathcal{V}}^p})$, we arrive at

$$\begin{aligned} &\pi\,(\dot{\bar{\delta}}^p - \dot{\delta}^p) + \sum_{i=1}^{3}\left[\sigma_i\,(\dot{\bar{\varepsilon}}_i^p - \dot{\varepsilon}_i^p) + \tau_i\,(\dot{\bar{\gamma}}_i^p - \dot{\gamma}_i^p)\right] \geq 0 \\ &\implies \pi\,(\dot{\delta}^p - \dot{\bar{\delta}}^p) + \sum_{i=1}^{3}\left[\sigma_i\,(\dot{\varepsilon}_i^p - \dot{\bar{\varepsilon}}_i^p) + \tau_i\,(\dot{\gamma}_i^p - \dot{\bar{\gamma}}_i^p)\right] \leq 0. \end{aligned} \tag{44}$$

Now the convex hull $\mathbb{C}_{\dot{\mathcal{V}}^p}$ of the set $l_{\dot{\mathcal{V}}^p}$ is given as

$$\mathbb{C}_{\dot{\mathcal{V}}^p} = \lambda_0\,\dot{\delta}^p + \sum_{i=1}^{3}\left(\lambda_i\,\dot{\varepsilon}_i^p + \lambda_{i+3}\,\dot{\gamma}_i^p\right) \;\text{ with }\; \lambda_0, \lambda_1, \ldots, \lambda_6 \geq 0 \;\text{ and }\; \sum_{i=0}^{6}\lambda_i = 1.$$

Clearly, the second formula in Eq. (44) implies that *the stress attributes l_σ are along the normal cone to the convex hull $\mathbb{C}_{\dot{\mathcal{V}}^p}$ at $\bar{l}_{\dot{\mathcal{V}}^p}$ in the plastic strain-rate space.* Recall that the plastic strain rates $\bar{l}_{\dot{\mathcal{V}}^p}$ correspond to the condition $Y(\bar{l}_{\dot{\mathcal{V}}^p}) = 1$. Geometrically, the convex hull $\mathbb{C}_{\dot{\mathcal{V}}^p}$ represents the set of all straight lines whose ends compose the set $l_{\dot{\mathcal{V}}^p}$. Eq. (44) implies that the stress attributes l_σ do not make an acute angle with any line segment $\mathbb{C}_{\dot{\mathcal{V}}^p}$ with $\bar{l}_{\dot{\mathcal{V}}^p}$ as endpoints for any set of values for $l_{\dot{\mathcal{V}}^p}$. Thus, l_σ is along the normal cone to the convex hull $\mathbb{C}_{\dot{\mathcal{V}}^p}$ at $\bar{l}_{\dot{\mathcal{V}}^p}$ (cf. Rockafellar, 1970, §2). Therefore, Eq. (44) acts as a normality rule for those materials that exhibit yielding behavior.

It is important to note that for materials that exhibit a creep-like behavior (and hence, have a smooth rate of dissipation function over any set of values for $l_{\dot{\mathcal{V}}^p}$), it is possible to derive a normality condition for each individual stress/plastic strain-rate pair (i.e., Eq. (41)). Such conditions, however, cannot be obtained for an individual stress/plastic strain-rate pair for materials that exhibit a yielding behavior. Much like the yield condition (40), for the latter case, the normality rule (44) involves all the stress/plastic strain-rate pairs.

Causality

In the above derivation, the stress attributes are held fixed, whereas the plastic strain-rate attributes are allowed to vary in order to maximize the rate of dissipation. Although the derived plastic flow rules are useful Ziegler (1963), the causality in Eq. (41) is reversed. Physically, the stress and plastic strain-rate attributes act as "determinants" and "resultants", respectively, in the terminology of Rajagopal and Srinivasa (2019). Note that with the assumed form of the Helmholtz potential function in Eq. (36), it is now possible to express the rate of dissipation function ξ and the reduced rate of dissipation constraint (38) in terms of the stress attributes l_σ.

For a given set of plastic strain-rate attributes $l_{\dot{\mathcal{V}}^p}$, the yield function can now be written as

$$Y(l_\sigma) = \max_{l_\sigma \neq \mathbf{0}} \frac{\pi\,\dot{\delta}^p + \sum_{i=1}^{3}\left(\sigma_i\,\dot{\varepsilon}_i^p + \tau_i\,\dot{\gamma}_i^p\right)}{\bar{\xi}(l_\sigma, l_{\dot{\mathcal{V}}^p})}. \tag{45}$$

It can easily be shown that the yield function is convex in the stress attributes space. (See Appendix for a detailed derivation.)

Now, maximization of the rate of dissipation function ξ with respect to l_σ yields

$$\dot{\delta}^p = \mu \frac{\partial \xi}{\partial \pi}, \qquad \dot{\varepsilon}_i^p = \mu \frac{\partial \xi}{\partial \sigma_i}, \qquad \dot{\gamma}_i^p = \mu \frac{\partial \xi}{\partial \tau_i} \tag{46}$$

where μ is the consistency parameter that satisfies the condition that $\mu = 0$ whenever $Y(l_\sigma) < 1$. The consistency parameter μ can be determined by substituting the plastic strain-rate attributes $l_{\dot{\mathcal{U}}^p}$ into the reduced rate of dissipation equation (38). Whenever μ is nonzero (implying, $\xi > 0$), it is possible to find nonzero values for the plastic strain-rate attributes. Thus, these conditions can be written as

$$\begin{aligned} &\mu = 0 \quad \text{whenever } Y(l_\sigma) < 1 \\ &\mu > 0 \quad \text{whenever } Y(l_\sigma) = 1. \end{aligned} \tag{47}$$

Eq. (47) is equivalent to the well-known consistency (KKT) condition used in classical plasticity theory.

From the above discussion, it can easily be understood that the set of Eqs. (41) acts as a dual of the flow rules (46). Note that in the derivations of both of these equations, the assumed form for the Helmholtz potential function plays a key role. In fact, even though it is possible to derive the flow rules (41) for some other form (Eq. (34)) without an assumption for the form of ψ, the same cannot be said about the flow rule (46). The assumed form for ψ is instrumental in the derivation of the latter. It is worth noting that the rate of dissipation function acts as a *plastic potential* in the flow rule (46). Following the arguments of Srinivasa (2010), one can easily show that whenever the rate of dissipation function ξ is a function of the plastic strain-rate attributes alone, then the yield function acts as a plastic potential, resulting in an associative flow rule. On the other hand, a non-associative flow rule emerges whenever the rate of dissipation is separable in terms of the functions of l_σ and $l_{\dot{\mathcal{U}}^p}$ (i.e., $\xi = \bar{\xi}(l_\sigma, l_{\dot{\mathcal{U}}^p}) = g(l_\sigma)\, h(l_{\dot{\mathcal{U}}^p})$) respectively. These cases will be demonstrated in Section 5 with the help of particular examples.

4. Volume-preserving plastic deformation

For metals and polymers, it is often assumed that the plastic deformation process is volume-preserving Lubliner (2008), i.e., $\det(\boldsymbol{\mathcal{U}}^p) = 1$, which further implies that the plastic dilatational strain δ^p and its rate are zero. However, materials like certain soils, rocks and foams exhibit a dilatant pressure-dependent elastoplastic behavior Srinivasa (2010) in which the volume of the natural configuration $\tilde{\kappa}_p$ does not remain a constant anymore. In this case, no other constraint in addition to the reduced rate of dissipation equation (38) is required. Therefore, the constitutive model developed so far is suitable for the latter class of materials. In this section, we show that the developed constitutive model can accommodate volume-preserving plastic deformation with slight modifications in the constrained optimization problem.

In the current theory, the assumption of a volume-preserving plastic deformation is manifested by considering the condition of a zero, plastic, dilatational, strain rate, $\dot{\delta}^p$, as an additional constraint, instead of using it as a kinematic variable.[4] In this case, the plastic dilatation term δ^p and its rate must be dropped from the argument of $\hat{\xi}$ in Eq. (28). Therefore, when the plastic deformation is assumed to be volume-preserving, the rate of dissipation function reduces to the form

$$\xi_{cv} = \hat{\xi}_{cv}\,(\varepsilon_i^p, \gamma_i^p, \dot{\varepsilon}_i^p, \dot{\gamma}_i^p) \tag{48}$$

where $i = 1, 2, 3$. Similarly, the Helmholtz potential function reduces to the form

$$\psi_{cv} = \hat{\psi}_{cv}(\delta, \varepsilon_i, \gamma_i, \varepsilon_i^p, \gamma_i^p). \tag{49}$$

Like before, we assume that the elastic response of the material is that of a Green elastic solid. Since the stress attributes depend only on the *total* strain attributes, they can be obtained from the Helmholtz potential function according to Eq. (31). Now using the definition for the rate of dissipation function (Eq. (27)), one can obtain a reduced rate of dissipation equation that, in this case, is

$$\hat{\xi}_{cv} = -\rho_0 \sum_{i=1}^{3} \left(\frac{\partial \psi}{\partial \varepsilon_i^p}\, \dot{\varepsilon}_i^p + \frac{\partial \psi}{\partial \gamma_i^p}\, \dot{\gamma}_i^p \right). \tag{50}$$

Note that the rate of dissipation function is no longer a function of the plastic dilatational strain rate $\dot{\delta}^p (= 0)$. Now, maximizing the rate of dissipation function $\hat{\xi}_{cv}$ with respect to the set of kinematic variables $\{\dot{\varepsilon}_i^p, \dot{\gamma}_i^p\}$, with Eq. (50) and $\dot{\delta}^p = 0$ as constraints, we finally obtain

$$\frac{\partial \hat{\xi}}{\partial \dot{\varepsilon}_i^p} = -\lambda\, \rho_0\, \frac{\partial \hat{\psi}}{\partial \varepsilon_i^p}; \qquad \frac{\partial \hat{\xi}}{\partial \dot{\gamma}_i^p} = -\lambda\, \rho_0\, \frac{\partial \hat{\psi}}{\partial \gamma_i^p} \tag{51}$$

where λ is a Lagrange multiplier. Eq. (51) has been derived by using the fact that the deformation modes, dilatation and squeeze, are independent of each other. This can also be corroborated by computing the partial derivative of the plastic strain-rate term with respect to any of the plastic, squeeze, strain rates (i.e., $\partial \dot{\delta}^p / \partial \dot{\varepsilon}_i^p = 0$). Now, following the procedure described in Section 3.5, one can

[4] We do not use the condition δ^p as a constraint, because the optimization is carried out only with respect to the plastic strain-rate attributes.

derive the yield and normality condition whenever a special form for the Helmholtz potential function is assumed. If the Helmholtz function, in this case, is assumed as

$$\begin{aligned}\psi = \hat{\psi}(l_{\mathcal{U}^e}, l_{\mathcal{U}^p}) = \frac{1}{2}\ \Big[\ N_{00}\,\delta^2 + \sum_{i=1}^{3} N_{0i}\,(\delta-\delta^p)\,(\varepsilon_i-\varepsilon_i^p) + \sum_{i=1}^{3} N_{0(i+3)}\,(\delta-\delta^p)\,(\gamma_i-\gamma_i^p) \\ + \sum_{\substack{i,j=1\\ i\le j}}^{3} N_{ij}\,(\varepsilon_i-\varepsilon_i^p)\,(\varepsilon_j-\varepsilon_j^p) + \sum_{i,j=1}^{3} N_{i(j+3)}\,(\varepsilon_i-\varepsilon_i^p)\,(\gamma_j-\gamma_j^p) \\ + \sum_{\substack{i,j=1\\ i\le j}}^{3} N_{(i+3)(j+3)}\,(\gamma_i-\gamma_i^p)\,(\gamma_j-\gamma_j^p)\ \Big],\end{aligned} \tag{52}$$

then the yield function can be written as

$$Y_{cv}(l_{\mathcal{U}^p}) := \max_{l_{\mathcal{U}^p}\neq\mathbf{0}} \frac{\sum_{i=1}^{3}\left(\sigma_i\,\dot{\varepsilon}_i^p + \tau_i\,\dot{\gamma}_i^p\right)}{\bar{\xi}(l_\sigma, l_{\mathcal{U}^p})}. \tag{53}$$

It can be easily shown that in this case the yield surface is convex in the plastic strain-rate (or stress) space. Finally, maximization of the rate of dissipation function yields

$$\sigma_i = \lambda\,\frac{\partial\xi}{\partial\dot{\varepsilon}_i^p}, \qquad \tau_i = \lambda\,\frac{\partial\xi}{\partial\dot{\gamma}_i^p}. \tag{54}$$

Eq. (54) provides an implicit equation for the plastic strain-rate attributes. Now, one can carry out the maximization by varying the stress attributes instead. In this case, the plastic strain-rate attributes can be obtained through the flow rule as

$$\dot{\varepsilon}_i^p = \mu\,\frac{\partial\xi}{\partial\sigma_i}, \qquad \dot{\gamma}_i^p = \mu\,\frac{\partial\xi}{\partial\tau_i} \tag{55}$$

where μ denotes the consistency parameter.

5. Illustration

With the constitutive model based on **QR** kinematics established, we now focus on some important examples. As mentioned earlier, a key advantage of using **QR** kinematics is an additive decomposition of the strain attributes into their respective elastic and plastic components. In general, such a decomposition is a key feature of the small-displacement gradient theory. However, in our case, an additive strain decomposition follows from the upper-triangular decomposition of the deformation gradient, even in the finite deformation setting. Therefore, we pick up some important models widely used in the small-strain plasticity theory and extend them into a finite deformation setting using our developed constitutive model.

5.1. J_2 Plasticity

The von Mises criterion[5] is possibly the most commonly used yield criterion for metal plasticity. According to this criterion, a material exhibits inelastic behavior whenever a quantity $\sqrt{J_2}$, associated with Cauchy stress components σ_{ij}, reaches the current yield stress in shear, i.e., k. The quantity J_2 is defined as

$$J_2 := \tfrac{1}{2}\,\mathrm{tr}(\hat{\boldsymbol{S}}\cdot\hat{\boldsymbol{S}}) = \tfrac{1}{2}\,\hat{S}_{ij}\,\hat{S}_{ij} \tag{56}$$

where $\hat{\boldsymbol{S}}$ is the symmetric deviatoric stress defined as

$$\hat{\boldsymbol{S}} = \boldsymbol{\sigma} - \tfrac{1}{3}\,\mathrm{tr}(\boldsymbol{\sigma})\,\boldsymbol{I}. \tag{57}$$

Therefore, a yield function can be written as

$$f(\boldsymbol{\sigma}) = J_2 - k^2. \tag{58}$$

The flow rule associated with this yield criterion is given by

$$\dot{\epsilon}_{ij}^p = \dot{\lambda}\,\frac{\partial f}{\partial\sigma_{ij}} \tag{59}$$

where ϵ_{ij}^p denote the plastic components of the strain tensor in a small displacement gradient theory.

In our theory, the constitutive assumption for two functions, namely, the Helmholtz potential function ψ and the rate of dissipation function ξ, must be specified at the beginning. Here we assume that the elastic response of the material is that of a

[5] This yield criterion and associated flow rule were first introduced by Lévy and later developed by von Mises. Their theory is applicable whenever the elastic strains are negligible. An extension of their theory to capture nonzero elastic strains was later proposed, the outcome being commonly known as the Prandtl–Reuss equations.

Green elastic solid. Therefore, the form for the Helmholtz potential function is same as that in Eq. (36). Note that J_2 plasticity results in an *associative* plastic flow rule. In order to obtain associative flow rules in our framework, the rate of dissipation function must be a function of the plastic strain-rate attributes alone. Because the J_2 theory was developed based on an assumption that plastic deformation is volume-preserving, we further assume that the plastic dilatational strain-rate $\dot{\delta}^p$ is zero. Let us choose a rate of dissipation function ξ of the form

$$\xi = k\sqrt{\sum_{i=1}^{3}\left((\dot{\varepsilon}_i^p)^2 + (\dot{\gamma}_i^p)^2\right)}. \tag{60}$$

With this assumed form for the rate of dissipation function ξ, while employing Eq. (37), the stress attributes can be obtained as

$$\sigma_i = \mu k\,\frac{\dot{\varepsilon}_i^p}{\sqrt{\sum_{i=1}^{3}\left((\dot{\varepsilon}_i^p)^2 + (\dot{\gamma}_i^p)^2\right)}} \quad \text{and} \quad \tau_i = \mu k\,\frac{\dot{\gamma}_i^p}{\sqrt{\sum_{i=1}^{3}\left((\dot{\varepsilon}_i^p)^2 + (\dot{\gamma}_i^p)^2\right)}}. \tag{61}$$

The consistency parameter μ is determined by satisfaction of the reduced rate of dissipation equation (50). Upon substituting the stress attributes from Eq. (61) into Eq. (50), we find that $\mu = 1$. In our theory, physically, the quantity J_2 is equivalent to

$$\mathcal{J}_2 := \sum_{i=1}^{3}\left(\sigma_i^2 + \tau_i^2\right). \tag{62}$$

In the classical J_2 theory, the independence of the yield function on the volumetric (or mean) stress is achieved by defining J_2 based on the deviatoric stress $\hat{\boldsymbol{S}}$, instead of the Cauchy stress $\boldsymbol{\sigma}$; whereas, in our theory, this independence is manifested by simply avoiding the volumetric stress π in the definition of $\mathcal{J}_2$. Substituting the expressions for stress attributes from Eq. (61), one can easily compute $\mathcal{J}_2$ as

$$\mathcal{J}_2 = \sum_{i=1}^{3}\left(\sigma_i^2 + \tau_i^2\right) = k^2 \implies \frac{\mathcal{J}_2}{k^2} = 1. \tag{63}$$

Now the yield function in this case is given as

$$Y = \frac{\sum_{i=1}^{3}(\sigma_i\,\dot{\varepsilon}_i + \tau_i\,\dot{\gamma}_i)}{\xi}. \tag{64}$$

By substituting the stress and plastic strain-rate attributes, and the assumed form for the rate of dissipation function ξ, one can easily show that the yield function Y is equal to the quantity $\mathcal{J}_2/k^2$. Therefore, Eq. (63) serves as the yield criterion in this case.

5.2. Drucker–Prager criterion

For many materials, like soils, rocks, foams, etc., the plastic deformation also depends on the volumetric stress π. To incorporate this in a plasticity model, Drucker and Prager (1952) came up with an extended version of the Mohr–Coulomb model and combined it with the von Mises yield criterion in such a way that the yield function f also depends upon the mean stress. Moreover, unlike the von Mises criterion, in this model, the plastic flow rules are derived from a separate plastic potential function G ($\neq f$), i.e., the flow rules are non-associative. In this model, the yield function is given as

$$f(\boldsymbol{\sigma}) := \sqrt{J_2} + \alpha\, p - k = 0 \tag{65}$$

whereas the plastic potential is given as

$$G(\boldsymbol{\sigma}) := \sqrt{J_2} + \beta p. \tag{66}$$

Here p is the mean stress, which is given as $p = \frac{1}{3}\,\mathrm{tr}(\boldsymbol{\sigma})$, while α and β are material parameters. This flow rule can be obtained from a plastic potential as

$$\dot{\boldsymbol{\epsilon}}^p = \dot{\lambda}\,\frac{\partial G(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}}. \tag{67}$$

To derive the Drucker–Prager model in our framework, we assume the same form for the Helmholtz potential function as in Eq. (36). The rate of dissipation function must be chosen in a way such that the plastic, volumetric, strain rate is also taken into account. Let us choose a rate of dissipation function ξ of the form

$$\xi(\pi, l_{\mathcal{V}^p}) = m_1(\pi)\sqrt{\sum_{i=1}^{3}\left((\dot{\varepsilon}_i^p)^2 + (\dot{\gamma}_i^p)^2\right)} + m_2(\pi)\,\frac{(\dot{\delta}^p)^2}{\sqrt{\sum_{i=1}^{3}\left((\dot{\varepsilon}_i^p)^2 + (\dot{\gamma}_i^p)^2\right)}}. \tag{68}$$

Now employing Eq. (51), the stress attributes are obtained as

$$\pi = \frac{2m_1(\pi)\dot{\delta}^p}{\sqrt{\sum_{i=1}^{3}\left((\dot{\varepsilon}_i^p)^2 + (\dot{\gamma}_i^p)^2\right)}}, \tag{69a}$$

$$\sigma_i = \left(m_1 - \frac{m_2\,(\dot{\delta}^p)^2}{\sum_{i=1}^{3}\left((\dot{\varepsilon}_i^p)^2 + (\dot{\gamma}_i^p)^2\right)} \right) \frac{\dot{\varepsilon}_i^p}{\sqrt{\sum_{i=1}^{3}\left((\dot{\varepsilon}_i^p)^2 + (\dot{\gamma}_i^p)^2\right)}}, \tag{69b}$$

$$\tau_i = \left(m_1 - \frac{m_2\,(\dot{\delta}^p)^2}{\sum_{i=1}^{3}\left((\dot{\varepsilon}_i^p)^2 + (\dot{\gamma}_i^p)^2\right)} \right) \frac{\dot{\gamma}_i^p}{\sqrt{\sum_{i=1}^{3}\left((\dot{\varepsilon}_i^p)^2 + (\dot{\gamma}_i^p)^2\right)}}. \tag{69c}$$

Now let us compute the equivalent Mises stress $\mathcal{J}_2$. Substituting the expressions for σ_i and τ_i from Eqs. (69b) and (69c), $\mathcal{J}_2$ can be computed as

$$\mathcal{J}_2 = \left(m_1(\pi) - m_2(\pi) \frac{(\dot{\delta}^p)^2}{\sum_{i=1}^{3}\left((\dot{\varepsilon}_i^p)^2 + (\dot{\gamma}_i^p)^2\right)} \right)^2. \tag{70}$$

Let us define the material dependent parameters k and α as $k = m_1$ and $\alpha(\pi) = \pi/4m_2(\pi)$. Note that parameter m_1 in Eq. (68) is equal to k, and thus, no longer needs to be a function of π. Therefore, if the material parameter k is not considered to be a function of π, then the Eq. (69a) provides an explicit expression for the dilatant pressure π. Substituting k and α in Eq. (70), one can write

$$\sqrt{\mathcal{J}_2} + \alpha\,\pi - k = 0 \implies \frac{\sqrt{\mathcal{J}_2} + \alpha\,\pi}{k} = 1. \tag{71}$$

From the definition of the yield function (40), one can easily show that the Eq. (71) acts as the yield condition in this case.

Therefore, we can conclude that with the proper choice of a Helmholtz potential ψ and a rate of dissipation function ξ, it is possible to recover classical plasticity models from our constitutive model, even in a finite deformation setting.

6. Conclusion

A novel constitutive model for elastic–plastic materials is developed using scalar, conjugate, stress/strain, base pairs arising from a **QR** decomposition of the deformation gradient. It has been shown that the multiplicative elastic–plastic decomposition of the Laplace stretch leads to an additive strain decomposition, which is commonly used in a small strain theory. This decomposition plays a key role in developing our constitutive model. In addition to the laws of thermodynamics, a maximum rate of dissipation criterion has been used to derive an evolution equation for the plastic strain rates. Two different models – one for a dilatant-pressure dependent deformation, and another for a volume-preserving plastic deformation – have been constructed. Finally, the examples of classic J_2 plasticity and of a compressible Drucker–Prager model have been demonstrated in our framework.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix. Convexity of yield function

Let us consider the case where the stress attributes are held constant and the plastic strain-rate attributes are allowed to vary in order to maximize the rate of dissipation function, ξ. Consider two sets of plastic strain-rate attributes $\bar{l}_{\dot{\mathcal{U}}^p} = \{\dot{\bar{\delta}}^p \quad \dot{\bar{\varepsilon}}_1^p \quad \dot{\bar{\varepsilon}}_2^p \quad \dot{\bar{\varepsilon}}_3^p \quad \dot{\bar{\gamma}}_1^p \quad \dot{\bar{\gamma}}_2^p \quad \dot{\bar{\gamma}}_3^p\}$ and $\hat{l}_{\dot{\mathcal{U}}^p} = \{\dot{\hat{\delta}}^p \quad \dot{\hat{\varepsilon}}_1^p \quad \dot{\hat{\varepsilon}}_2^p \quad \dot{\hat{\varepsilon}}_3^p \quad \dot{\hat{\gamma}}_1^p \quad \dot{\hat{\gamma}}_2^p \quad \dot{\hat{\gamma}}_3^p\}$ along with a prescribed set of values for the stress attributes $\bar{l}_\sigma = \{\bar{\pi} \quad \bar{\sigma}_1 \quad \bar{\sigma}_2 \quad \bar{\sigma}_3 \quad \bar{\tau}_1 \quad \bar{\tau}_2 \quad \bar{\tau}_3\}$. Now, from the definition of our yield function (40), we can say that for any arbitrary plastic strain-rate attributes $\bar{l}_{\dot{\mathcal{U}}^p}$ and $\hat{l}_{\dot{\mathcal{U}}^p}$, the following conditions hold:

$$\frac{\bar{\pi}\,\dot{\bar{\delta}}^p + \bar{\sigma}_1\,\dot{\bar{\varepsilon}}_1^p + \bar{\sigma}_2\,\dot{\bar{\varepsilon}}_2^p + \bar{\sigma}_3\,\dot{\bar{\varepsilon}}_3^p + \bar{\tau}_1\,\dot{\bar{\gamma}}_1^p + \bar{\tau}_2\,\dot{\bar{\gamma}}_2^p + \bar{\tau}_3\,\dot{\bar{\gamma}}_3^p}{\bar{\xi}(\bar{l}_{\dot{\mathcal{U}}^p})} \le Y(\bar{l}_{\dot{\mathcal{U}}^p}) \tag{A.1a}$$

and

$$\frac{\bar{\pi}\,\dot{\hat{\delta}}^p + \bar{\sigma}_1\,\dot{\hat{\varepsilon}}_1^p + \bar{\sigma}_2\,\dot{\hat{\varepsilon}}_2^p + \bar{\sigma}_3\,\dot{\hat{\varepsilon}}_3^p + \bar{\tau}_1\,\dot{\hat{\gamma}}_1^p + \bar{\tau}_2\,\dot{\hat{\gamma}}_2^p + \bar{\tau}_3\,\dot{\hat{\gamma}}_3^p}{\bar{\xi}(\bar{l}_{\dot{\mathcal{U}}^p})} \le Y(\hat{l}_{\dot{\mathcal{U}}^p}). \tag{A.1b}$$

The yield function for the plastic strain-rate attributes

$$\bar{l}_{\dot{\mathcal{U}}^p} + \hat{l}_{\dot{\mathcal{U}}^p} = \{\dot{\bar{\delta}}^p + \dot{\hat{\delta}}^p \quad \dot{\bar{\varepsilon}}_1^p + \dot{\hat{\varepsilon}}_1^p \quad \dot{\bar{\varepsilon}}_2^p + \dot{\hat{\varepsilon}}_2^p \quad \dot{\bar{\varepsilon}}_3^p + \dot{\hat{\varepsilon}}_3^p \quad \dot{\bar{\gamma}}_1^p + \dot{\hat{\gamma}}_1^p \quad \dot{\bar{\gamma}}_2^p + \dot{\hat{\gamma}}_2^p \quad \dot{\bar{\gamma}}_3^p + \dot{\hat{\gamma}}_3^p\}$$

can be written in the same manner as

$$\begin{aligned} Y(\bar{l}_{\dot{\mathcal{U}}^p} + \hat{l}_{\dot{\mathcal{U}}^p}) &= \max_{\bar{l}_{\dot{\mathcal{U}}^p} \neq \mathbf{0}} \frac{(\dot{\bar{\delta}}^p + \dot{\hat{\delta}}^p)\,\bar{\pi} + \sum_{i=1}^{3}\left((\dot{\bar{\varepsilon}}_i^p + \dot{\hat{\varepsilon}}_i^p)\,\bar{\sigma}_i + (\dot{\bar{\gamma}}_i^p + \dot{\hat{\gamma}}_i^p)\,\bar{\tau}_i\right)}{\bar{\xi}(\bar{l}_{\dot{\mathcal{U}}^p})} \\ &= \max_{\bar{l}_{\dot{\mathcal{U}}^p} \neq \mathbf{0}} \frac{\dot{\bar{\delta}}^p\,\bar{\pi} + \sum_{i=1}^{3}\left(\dot{\bar{\varepsilon}}_i^p\,\bar{\sigma}_i + \dot{\bar{\gamma}}_i^p\,\bar{\tau}_i\right)}{\bar{\xi}(\bar{l}_{\dot{\mathcal{U}}^p})} + \frac{\dot{\hat{\delta}}^p\,\bar{\pi} + \sum_{i=1}^{3}\left(\dot{\hat{\varepsilon}}_i^p\,\bar{\sigma}_i + \dot{\hat{\gamma}}_i^p\,\bar{\tau}_i\right)}{\bar{\xi}(\bar{l}_{\dot{\mathcal{U}}^p})} \le Y(\bar{l}_{\dot{\mathcal{U}}^p}) + Y(\hat{l}_{\dot{\mathcal{U}}^p}). \end{aligned} \tag{A.2}$$

The last of Eq. (A.2) is derived by using Eq. (A.1). Therefore, from Eq. (A.2), we conclude that *the yield function is convex in the plastic strain-rate space.*

Now, alternatively, one can choose to allow the stress attributes to vary while keeping the plastic strain rates fixed. In that case, if we consider two different sets of values for the stress attributes $\bar{l}_\sigma$ and $\hat{l}_\sigma$ with a fixed set of plastic strain-rate attributes $\bar{l}_{\mathring{U}^p}$, then following a similar procedure, we can readily conclude that *the yield function is also convex in stress space.*

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